Definition 0.1. $\mathcal{O} = \wp(\mathcal{G})$. Can also be, e.g., finite subsets, or compact subsets. △

1 Worse-than

Instead of directly axiomatising rejection functions, with $R(A) \subseteq A$, we will axiomatise a seemingly more general notion, which we might think of "I would reject it were it to be available"

Here's some other potential readings of $g \in W(A)$:

- g is impermissible to choose from $A \cup \{g\}$: $g \in R(A \cup \{g\})$
- g is dis-preferred to something in A.
  This seems to have two different and incompatible readings when we're supervaluating over probabilities (or total comparative orders)
  - There is something in A which I (det) prefer to g (Max)
  - It is determinately true that there's something which is preferable (E-Ad)

  I'm not sure what a unifying underlying reading of this is.
- Supervaluating over a collection of comparative orders on $\mathcal{G}$, which might themselves be partial, and that there is some $f \in A$ such that $g \prec f$.
  - But I wonder when we're constructing a model of someone's opinions when it's right to put an incomparability into a determinate partial comparative vs indeterminate what the total comparative is.

Main difference: not restricted to the members of A themselves. In fact, contains any gambles weakly dominated by something in A. For the "epistemic" interpretations, that's entirely natural, for the choice-y ones, it's just "add g".

Is this the right name? Kevin calls $g \in W(A)$ "set preference", written $g < A$.

Definition 1.1. $W : \mathcal{O} \to \mathcal{O}$

($\mathrm{W}_{\setminus}$)

$$h \in W(A) \implies h \in W(A \setminus \{h\}).$$

This is equivalent (given the other axioms[1]) to Aizerman's.

$$f \in W(A) \text{ and } h \in W(A) \implies h \in W(A \setminus \{f\}).$$

[2]

[1] recentre, scalar and convex; or rather,

[2] Thanks to Jasper, Arne and Arthur (via Arthur) for discussion and that we worked out that this is the core additional content that that axiom adds on top of the remaining axioms.

(W$_\varnothing$)

$$h \notin W(\varnothing)$$

(W$_\supseteq$) Suppose $A \subseteq B$, then

$$h \in W(A) \implies h \in W(B)$$

(W$_{\text{conv}}$) Suppose that for every $g_1 \in A_1, g_2 \in A_2$, $f_{\langle g_1, g_2\rangle} \in \mathsf{conv}(\{g_1, g_2\})$. Then

$$h \in W(A_1) \text{ and } h \in W(A_2) \implies h \in W(\{f_{\langle g_1,g_2\rangle} \mid \langle g_1, g_2\rangle \in A_1 \times A_2\})$$

Note: we can of course then extend this up to finite sets $A_1, \ldots, A_n$. Maybe we should directly state it that way, but I'm trying to stay close to a simple K axiomatisation.

If we extend it to infinitely many simultaneously then we get representation by a set of orders rather than a filter of them.

Also note: we replace the normal addition axiom of rejection functions with this, because the addition axiom is only valid when it's the 0 gamble which we're comparing too, whereas the conv one is valid for arbitrary h. The addition one then follows for 0 by the conv (take half-half mixture)and scalar.

(W$_{\text{scalar}}$) Suppose that for every $g \in A$, λ_g is a positive real value (scalar). Then

$$0 \in W(A) \implies 0 \in W(\{\lambda_g g \mid g \in A\}).$$

(W$_{\text{translation}}$) Let g be any member of $\mathcal{G}$ and $A + g := \{f + g \mid f \in A\}$. Then

$$h \in W(A) \implies h + g \in W(A + g)$$

(W$_{\gneqq}$) Suppose $g \lneqq f$. Then $g \in W(\{f\})$

(W$_{\geqslant}$) Suppose that for each $g \in A$, $f_g \geqslant g$, then

$$h \in W(A) \implies h \in W(\{f_g \mid g \in A\}).$$

I think this follows from the remaining when Ω is finite, or option sets must be finite, or perhaps also when we have some inf conv axiom. Not sure about the general case.

$\triangle$

Remark 1.2 (Finite option sets)**.** Then $\geqslant$ can be deleted.

Also Conv and Scalar can be replaced by one gamble at a time versions. They can then just be (by induction) all replaced to get the general versions. This is what Arne and Jasper do.

($\mathrm{W}_{\mathrm{conv}}$) Suppose $f \in \mathsf{conv}(\{g_1, g_2\})$. Then

$$h \in W(A \cup \{g_1\}) \text{ and } h \in W(A \cup \{g_2\}) \implies h \in W(A \cup \{f\})$$

($\mathrm{W}_{\mathrm{scalar}}$) Suppose that λ is a positive real value (scalar). Then

$$0 \in W(A \cup \{g\}) \implies 0 \in W(A \cup \{\lambda g\}).$$

$\triangle$

Remark 1.3 (scalar axiom)**.** Note that to make the scalar axiom not particular for 0, we'd have to use:

$$h \in W(A) \implies h \in W(\{h + \lambda_g (g - h) \mid g \in A\}).$$

This is invalid:

$$h \in W(A) \implies h \in W(\{\lambda_g g \mid g \in A\}).$$

This one is valid:

$$h \in W(A) \implies \lambda h \in W(\{\lambda g \mid g \in A\}).$$

But it's weaker.

Conv, however, does not need to be so-modified because convex combinations are translation-invariant anyway. This is why we use Conv not Add. $\triangle$

Proposition 1.4. *Γ-maximin with a uniformly regular set of probabilities satisfies everything except Translation. It does, however satisfy*

($\mathrm{W}_{\text{const-translation}}$) *Let c be a constant gamble and $A + c := \{f + c \mid f \in A\}$. Then*

$$h \in W(A) \implies h + c \in W(A + c).$$

Additional optional axioms:

($\mathrm{W}_{\mathrm{EAd/Mixing}}$)

$$h \in W(\mathsf{closure}(A)) \implies h \in W(A) \tag{1}$$

$$h \in W(\mathsf{conv}(A)) \implies h \in W(A) \tag{2}$$

($\mathrm{W}_{\mathrm{Maximality/Binarification}}$) If $f \in W(A)$ then there is some $g \in A$ st $f \in W(\{g\})$

($\mathrm{W}_{\mathrm{InfiniteClosure}}$) What axiom gets us principal filters, i.e., a single set of dominating probabilities/orders? I.e., in the E-Ad case, a credal set rather than probability filter?

Infinite version of Conv axiom? I'd prefer something separated from the idea of convexity, though.

Suppose $(A_i)_{i \in I}$ is a family of option sets and for every choice of $\sigma \in \bigtimes_{i \in I} A_i$, we have $f_\sigma \in \mathsf{conv}\{\sigma(i) \mid i \in I\}$. Then

$$[h \in W(A_i) \text{ for all } i \in I] \implies h \in W\left(\{f_\sigma \mid \sigma \in \bigtimes_{i \in I} A_i\}\right).$$

($\mathrm{W}_{\mathrm{InfiniteClosureViaPicking}}$) Is this one equivalent? It's valid, but not sure it entails the previous one. I've just got rid of 'conv'.

Suppose $(A_i)_{i\in I}$ is a family of option sets and for every choice of $\sigma \in \bigtimes_{i\in I} A_i$, we have $f_\sigma \in \{\sigma(i) \mid i \in I\}$. Then

$$[h \in W(A_i) \text{ for all } i \in I] \implies h \in W\left(\{f_\sigma \mid \sigma \in \bigtimes_{i\in I} A_i\}\right).$$

2 Rejection axioms

Interpretation of R: g is impermissible to choose from option set A

Not sure if I prever conv axiom v1 or v2

Definition 2.1. $R : \mathcal{O} \to \mathcal{O}$

($\mathrm{R}_{\mathrm{rej}}$)

$$R(A) \subseteq A.$$

($\mathrm{R}_{\varnothing}$)

$$f \notin R(\{f\}).$$

($\mathrm{R}_{\supseteq}$) Suppose $A \subseteq B$. Then

$$h \in R(A) \implies h \in R(B).$$

($\mathrm{R}_{\gneqq}$) Suppose $g \lneqq f$. Then

$$f, g \in A \implies g \in R(A).$$

($\mathrm{R}_{\geqslant}$) Suppose that for each $g \in A$, $f_g \geqslant g$. Then

$$h \in R(A) \implies h \in R(\{f_g \mid g \in A\}).$$

($\mathrm{R}_{\mathrm{conv-v1}}$) Suppose that for every $g_1 \in A_1$ and $g_2 \in A_2$, $f_{\langle g_1,g_2\rangle} \in \mathsf{conv}(\{g_1, g_2\})$. Then

$$\begin{aligned} &h \in R(A_1) \cap R(A_2) \text{ and } h \in \{f_{\langle g_1,g_2\rangle} \mid \langle g_1, g_2\rangle \in A_1 \times A_2\} \\ &\qquad \implies h \in R\left(\{f_{\langle g_1,g_2\rangle} \mid \langle g_1, g_2\rangle \in A_1 \times A_2\}\right). \end{aligned}$$

($\mathrm{R}_{\mathrm{conv-v2}}$) Suppose that for every $g_1 \in A_1$ and $g_2 \in A_2$, $f_{\langle g_1,g_2\rangle} \in \mathsf{conv}(\{g_1, g_2\})$. Then

$$\begin{aligned} &h \in R(A_1 \cup \{h\}) \cap R(A_2 \cup \{h\}) \\ &\qquad \implies h \in R\left(\{f_{\langle g_1,g_2\rangle} \mid \langle g_1, g_2\rangle \in A_1 \times A_2\} \cup \{h\}\right). \end{aligned}$$

Note: we can of course then extend this up to finite sets $A_1, \dots, A_n$.

(R_{scalar}) Suppose that for every $g \in A$, λ_g is a positive real value. Then

$$0 \in R(A) \implies 0 \in R(\{\lambda_g g \mid g \in A\}).$$

(As $0 \in R(A) \subseteq A$, it is automatically in the scaled version.)

($R_{\text{translation}}$) Let g be any member of $\mathcal{G}$ and $A + g := \{f + g \mid f \in A\}$. Then

$$h \in R(A) \implies h + g \in R(A + g).$$

$\triangle$

Conjecture 2.2. *The two foramlisms are interderivable:*
And the axioms are then eqivalent.

Note: the \ axiom has gone because it follows from the others.[3]

Conjecture 2.3. *When A is compact, these imply $R(A) \neq A$.*

Proof Attempt. content... $\triangle$

3 Equiv to K

$\mathcal{K}$ are gamble sets which contain something at least as good as 0: where $0 \in W(A)$.

Definition 3.1. $\mathcal{K} \subseteq \wp(\mathcal{G})$ is *coherent* if it satisfies

($K_\varnothing$) $\varnothing \notin \mathcal{K}$

($K_\supseteq$) If $A \in \mathcal{K}$ and $B \supseteq A$, then $B \in \mathcal{K}$

($K_{\setminus\{0\}}$) If $A \in \mathcal{K}$ then $A \setminus \{0\} \in \mathcal{K}$.

($K_{\gneq 0}$) If $g \in \mathcal{G}_{\gneq 0}$, then $\{g\} \in \mathcal{K}$.

($K_\geqslant$) Suppose that for each $g \in A$, f_g is some gamble where $f_g \geqslant g$, Then

$$A \in \mathcal{K} \implies \{f_g \mid g \in A\} \in \mathcal{K}$$

(K_{conv}) Suppose that for every $g_1 \in A_1$ and $g_2 \in A_2$, $f_{\langle g_1, g_2\rangle} \in \mathsf{conv}(\{g_1, g_2\})$. Then

$$A_1, A_2 \in \mathcal{K} \implies \{f_{\langle g_1, g_2\rangle} \mid \langle g_1, g_2\rangle \in A_1 \times A_2\} \in \mathcal{K}$$

(K_{scalar}) Suppose that for every $g \in A$, λ_g is a positive real value. Then

$$A \in \mathcal{K} \implies \{\lambda_g g \mid g \in A\} \in \mathcal{K}.$$

[3]Arthur.

$\triangle$

Definition 3.2. • $A \in \mathcal{K}_W$ iff $0 \in W(A)$

- $f \in W_{\mathcal{K}}(A)$ iff $A - f \in \mathcal{K}$

$\triangle$

W has structure that $\mathcal{K}$ doesn't have, if recentre fails, for example Γ-maximin. All the structure of this $\mathcal{K}$, however, is derivable from W.

Conjecture 3.3. *W satisfying* **??** *are expressively equivalent to K*

Conjecture 3.4. *coherence axioms on $\mathcal{K}$ and W are equivalent.*

Part I
E-admissibility

4 underlying $\succ$ are strict weak orders

4.1 Comparative

Proposition 4.1. *Given*

($\prec_{\text{recentre}}$) *If $f \succ g$, then $f + h \succ g + h$.*

($\prec_{\text{scalar}}$) *If $f \succ g$, then $\lambda f \succ \lambda g$ for every $\lambda > 0$.*

the following axioms are equivalent:

($\prec_{\text{mix}}$) *If $\alpha f + (1-\alpha)g \succ h$, where $0 < \alpha < 1$, then either $f \succ h$ or $g \succ h$.*

($\prec_{\text{add}}$) *If $f + g \succ 0$, then either $f \succ 0$ or $g \succ 0$.*

($\prec_{\text{negtrans}}$) *If $f \prec g$, then either $f \prec h$ or $h \prec g$.*

($\prec_{\text{transofincomp}}$) *The absence of $\prec$ is transitive.*

Proof. [4] We first use scalar to show that mix is equivalent to add: Indeed, if $f + g \succ 0$, then by scalar,

$$^1\!/_2 f + {}^1\!/_2 g \succ 0.$$

Hence ($\prec_{\text{mix}}$) gives $f \succ 0$ or $g \succ 0$.

Conversely, suppose

$$\alpha f + (1-\alpha)g \succ 0.$$

Then this is the sum of αf and $(1-\alpha)g$. By the additive form,

$$\alpha f \succ 0 \quad \text{or} \quad (1-\alpha)g \succ 0.$$

[4] proof written by ChatGPT

Since $\alpha > 0$ and $1 - \alpha > 0$, scalar gives

$$f \succ 0 \quad \text{or} \quad g \succ 0.$$

We now show that the additive form implies negative transitivity. Suppose

$$f \prec g.$$

By recentring, this is equivalent to

$$g - f \succ 0.$$

But

$$g - f = (g - h) + (h - f).$$

By the additive form, either

$$g - h \succ 0 \quad \text{or} \quad h - f \succ 0.$$

Using recentring again, these are respectively equivalent to

$$h \prec g \quad \text{or} \quad f \prec h.$$

Therefore

$$f \prec g \quad \Longrightarrow \quad h \prec g \text{ or } f \prec h.$$

Conversely, assume negative transitivity. Suppose

$$f + g \succ 0.$$

By recentring, this is equivalent to

$$-g \prec f.$$

Apply negative transitivity with $h = 0$. Then either

$$0 \prec f \quad \text{or} \quad -g \prec 0.$$

The first says

$$f \succ 0.$$

The second, by recentring with g, says

$$0 \prec g,$$

hence

$$g \succ 0.$$

So

$$f + g \succ 0 \quad \Longrightarrow \quad f \succ 0 \text{ or } g \succ 0.$$

□

Proposition 4.2. *$\prec$ is neg-trans, refl and trans iff it is the strict part of a total preorder.*

https://en.wikipedia.org/wiki/Weak_ordering#Strict_weak_orderings

4.2 Stating for R

Arbitrarily choose gambles $h_g \in \mathcal{G}$ for each $g \in A$. Consider:

($\mathrm{K}_{\mathrm{split}}$) For every $g \in A$, let h_g be any member of $\mathcal{G}$

$$A \in \mathcal{K} \implies \{h_g,\ g - h_g \mid g \in A\} \in \mathcal{K}.$$

The idea is that we have arbitrarily split each g into

$$g = h_g + (g - h_g).$$

So if $g \succ 0$, then at least one of the two pieces must be $\succ 0$, by the additive version of mix.

We could also phrase this axiom as

($\mathrm{K}_{\mathrm{split2}}$) Suppose that h_g^1, h_g^2 are such that $g = h_g^1 + h_g^2$. Then

$$A \in \mathcal{K} \implies \{h_g^1, h_g^2 \in A\} \in \mathcal{K}.$$

Using scalings, we should similarly get the symmetric version:

$$A \in \mathcal{K} \implies \{g - h_g,\ g + h_g \mid g \in A\} \in \mathcal{K}.$$

Here the idea is that

$$g = {}^1\!/\!{}_2(g - h_g) + {}^1\!/\!{}_2(g + h_g).$$

So, by the usual mix axiom, if $g \succ 0$, then at least one of

$$g - h_g, \qquad g + h_g$$

must be $\succ 0$.

Thus this is just a special case of the usual mixing axiom, but written in a useful split form.

What about the corresponding transitivity form? For a comparative order, the relevant axiom is

$$f \prec g \implies f \prec h \text{ or } h \prec g.$$

The corresponding W-version is:

($\mathrm{W}_{\mathrm{split1}}$) h_g are arbitrary

$$f \in W(A) \implies f \in W(\{h_g,\ f + g - h_g \mid g \in A\}).$$

The idea is: if $f \in W(A)$, then for each relevant order there is some $g \in A$ with

$$f \prec g.$$

Applying the transitivity-style axiom to $f \prec g$ and the arbitrary comparison point h_g, we get

$$f \prec h_g \quad \text{or} \quad h_g \prec g.$$

The first disjunct already says that h_g beats f. But W needs everything compared to the same base point, in this case we're using f. So to express the second disjunct, we use

$$f \prec f + g - h_g.$$

This is why the set in the W-axiom has the slightly odd-looking form

$$\{h_g,\ f + g - h_g \mid g \in A\}.$$

The term $f + g - h_g$ is just the shifted version of $h_g \prec g$.

($\mathrm{W}_{\mathrm{split2}}$) Suppose that h_g^1, h_g^2 are such that $g = h_g^1 + h_g^2$. Then

$$g = f + h_g^1 + h_g^2$$

Then

$$f \in W(A) \implies f \in W(\{f + h_g^1,\ f + h_g^2 \mid g \in A\}).$$

Proposition 4.3. *Assume $\mathcal{K}$ satisfies the scalar axiom. Then the following are equivalent for finite A.*

($\mathrm{K}_{\mathrm{split}}$) *For every $A \in \mathcal{K}$ and every choice of gambles $h_g \in \mathcal{G}$ for $g \in A$,*

$$\{h_g,\ g - h_g \mid g \in A\} \in \mathcal{K}.$$

($\mathrm{K}_{\mathrm{conv\text{-}pullback}}$) *If $A \in \mathcal{K}$ and $A \subseteq \mathsf{conv}(B)$, then*

$$B \in \mathcal{K}.$$

Proof. First suppose $(K_{\mathrm{conv\text{-}pullback}})$. Let $A \in \mathcal{K}$, and choose gambles h_g for each $g \in A$. By scalar,

$$\frac{1}{2}A := \left\{\frac{1}{2}g \mid g \in A\right\} \in \mathcal{K}.$$

But for every $g \in A$,

$$\frac{1}{2}g = \frac{1}{2}h_g + \frac{1}{2}(g - h_g),$$

so

$$\frac{1}{2}A \subseteq \mathsf{conv}\{h_g,\ g - h_g \mid g \in A\}.$$

Hence, by $(K_{\mathrm{conv\text{-}pullback}})$,

$$\{h_g,\ g - h_g \mid g \in A\} \in \mathcal{K}.$$

So we get (K_{split}).

Conversely, suppose (K_{split}). We first derive the finite mixing version. Suppose $B \in \mathcal{K}$, and for every $b \in B$ we have a finite convex decomposition

$$b = \sum_{i=1}^{n_b} \lambda_i^b a_i^b,$$

where each $\lambda_i^b > 0$, $\sum_i \lambda_i^b = 1$, and $a_i^b \in A$.

Using (K_{split}) repeatedly, split each b into the pieces

$$\lambda_1^b a_1^b, \quad \lambda_2^b a_2^b, \quad \ldots, \quad \lambda_{n_b}^b a_{n_b}^b.$$

At each step we are just using the identity

$$x = y + (x - y).$$

Therefore

$$\{\lambda_i^b a_i^b \mid b \in B,\ 1 \leqslant i \leqslant n_b\} \in \mathcal{K}.$$

By the scalar axiom, we may divide each element by its positive coefficient λ_i^b, so

$$\{a_i^b \mid b \in B,\ 1 \leqslant i \leqslant n_b\} \in \mathcal{K}.$$

This set is a subset of A. Hence, by upward closure,

$$A \in \mathcal{K}.$$

Thus, whenever $B \in \mathcal{K}$ and every member of B is a convex combination of members of A, we get $A \in \mathcal{K}$. That is,

$$B \in \mathcal{K} \text{ and } B \subseteq \mathsf{conv}(A) \quad \Longrightarrow \quad A \in \mathcal{K}.$$

$\square$

Not sure the equivalence applies to infinite sets since each may have a different numbering of splits and this is a step-by-step process so not sure how we do unboundedly many splits "simultaneously"

If it doesn't go to the infinite setting, here's an axiom we could use:

($\mathrm{K}_{mix-split}$) If each $g = h_g^1 + \ldots + h_g^{n_g}$ then

$$A \in K \implies (\{h_g^1, \ldots, h_g^{n_g} \mid g \in A\}) \in \mathcal{K}$$

Intuition, if g can be split into the pieces $g = h_g^1 + \ldots + h_g^{n_g}$ then at least one of those pieces must be positive. Note that it needs to be properly shifted to compare to f. Use

($\mathrm{W}_{mix-split}$) If each $g = f + (h_g^1 + \ldots + h_g^{n_g})$ then

$$f \in W(A) \implies f \in W(\{f + h_g^1,\ \ldots,\ f + h_g^{n_g} \mid g \in A\})$$

Or, alternatvely, use the conv form

(W$_{mix-conv}$) If each $g = \lambda_g^1 h_g^1 + \ldots + \lambda_g^{n_g} h_g^{n_g}$, in the conv. then

$$f \in W(A) \implies f \in W(\{h_g^1, \ldots, h_g^{n_g} \mid g \in A\})$$

Intuition, if $f \prec g$ and g can be split into the pieces $g = h_g^1 + \ldots + h_g^{n_g}$ then at least one of those pieces must be $\succ f$, as its a linear order.

5 Archimedeanity - Openness of generating $\succ$

Proposition 5.1. *Say $\prec$ is open iff: if $f \prec g$ then there is some $\epsilon > 0$ with $f \prec g - \epsilon$*

That is, the D it corresponds to is an open set.

Proposition 5.2. *W is generated by a set of orders each of which is open iff:*

(W$_{open}$) *If $f \in W(A)$ then $f \in W(\{g - \epsilon \mid g \in A, \epsilon > 0\})$*

(W$_{open}$) *If $f \in W(A)$ then and h_g^n is a sequence converging topologically to g, then $f \in W(\{h_g^n \mid g \in A, n > 0\})$*

(K$_{open}$) *If $A \in K$ then $\{g - \epsilon \mid g \in A, \epsilon > 0\} \in K$*

Note, that these two are the "mix-conv" and "mix-cl" from 4.5 of `https://www.ccampbell-moore.com/papers/Probability%20Filters%20IJAR.pdf`

6 E-Admissibility

Proposition 6.1 (?). *W is generated from a probability filter with E-admissibility iff it satisfies those axioms, including open and mixing.*

It is generated from a credal set with E-admissibility iff it also satisfies the infinite convex axiom.