Probability Filters and Desirable Gambles

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Abstract

In this paper we propose a mathematical model for imprecise probability representing an agent's uncertain beliefs. The proposed model is closely related to the credal set model, which uses a set of probability functions. The credal set model can be conceived of as encoding judgements such as it seeming more likely that it'll rain than snow. It requires that these judgements be closed under consequences according to the probability functions that satisfy the judgements. We adopt a similar idea but instead only require her judgements to be closed under finite consequences according the probability functions satisfying them. The formal model that we will provide for this is a collection of probability constraints which is closed under finite intersection and supersets. That is, it forms the mathematical structure of a filter. We will show how this allows the model to avoid some criticisms of the credal set model by Walley which led him to instead arguing for a model of uncertainty as a set of desirable gambles directly. We will show a close connection between this desirable gambles model and the proposed model of probability filters.

1 Introduction

In work on imprecise probabilities, various mathematical models are provided to capture an agent's uncertain belief state (see especially [68]). In this paper we develop and discuss an alternative mathematical model closely related to the very prominent approach of representing an agent's belief state, or credal state, with a set of probability functions, often called the credal set model [e.g., 36, 37, 20, 28, 65, 2, 50, 3, 42, 21].

A credal set is sometimes taken to merely be a formal representor of some other, more primitive, kinds of judgements of the agent reflecting her belief state. For example, one might consider a judgement expressed by "it seems more likely to rain than to snow". Such judgements can be seen to be satisfied by some probability functions and not by others. For example, this judgement is satisfied by those functions assigning a higher probability to it raining than to it snowing.

*These ideas were initially developed in joint work with Jason Konek [7]. I am very grateful to him as well as to Arthur van Camp, Kevin Blackwell, Gert de Cooman, Giacomo Molinari and Teddy Seidenfeld for very helpful discussions. The referees of ISIPTA and IJAR have provided detailed and helpful comments which has vastly improved the paper. I am very grateful to them. Some of my work was funded as a Leverhulme research fellow.

In this paper, we propose modelling one's belief state with a collection of judgements, each of which is satisfied by some probability functions. We will include the judgements that the agent explicitly holds, along with those that she is committed to because they "follow from" her explicitly held judgements.[1] The credal set model is recovered when one characterises "follows from" by entailment in accordance with finitely additive probability theory: If every finitely-additive probability function satisfying all her explicitly held judgements has another property, then this further probability constraint is something she is also committed to. For example, if she holds the judgement that it is 0.8-likely that Jones smokes, then she is committed by coherence to the judgement that it is 0.2-likely that Jones does not smoke.[2]

This is essentially the account by Van Fraassen [63, 64, 65, 66][3] and is close to some comments by Joyce [28, 26, 24]. For Van Fraassen, the agent's state of opinion is characterised by a collection of judgements and coherence is specified by entailment according to probability functions satisfying the judgements. A credal set is merely a "representor" of coherent collections of judgements. This model is also essentially proposed by Moss [44], which formed the origin of this paper, as joint work with Jason Konek [7].

This offers a mathematically equivalent reformulation of the credal set model and is compatible with many accounts of credal sets. Instead of focusing on a single set of probability functions, one focuses on the properties that the probability functions in the set have in common. One can specify a credal set and read off such judgements, or start off with judgements and read off a collection of probability functions satisfying them all. If one's judgements are closed and consistent under entailment according to the probability functions satisfying them then the two accounts are mathematically equivalent.

However, at this point we make an important alteration to the model: We will restrict the notion of entailment that we impose for coherence to only consider finitely many judgements at a time.[4] If a judgement follows from finitely many of her explicitly held judgements, in the sense that every (finitely-additive) probability function satisfying the held judgements also satisfies the further judgement, then we say that she is committed to the further judgement. But if it follows only when infinitely many of her explicitly held judgements are considered together, then we do not say she is committed to the further judgement.

We will call this the probability filter model of belief, as the structure of the probability constraints she is committed to forms the mathematical structure of a filter on probability functions.

This might be motivated by considerations on the limitations of reasoning we can hold a coherent agent to. Perhaps she is unable to survey her infinitely many judgements simultaneously. The coherence notion that we impose only constrains finite subsets of the agent's judgements at a time. This move bears some connection to responses to arguments for countable additivity, such as countable Dutch book arguments [17, 47]. This allows her judgements to be infinitely incompatible: there may in fact be no probability function which satisfies her infinitely many judgements simultaneously.

[1]Thanks to a referee for encouraging this separation. It also is adopted by Van Fraassen and highlighted in much work on natural extensions in the imprecise probability literature.

[2]See also Moss [44, §1.1] for this example.

[3]In particular, the sections: [63, §6-§7], [64, p.250-252] [65, p.345-347], [66, p.483-485].

[4]That is, we ensure that it is a *compact* notion of entailment.

It allows, for example, that an agent may judge the expected value of a given gamble to be positive but nonetheless, for any small price she would have to pay, she judges it to have a negative expected value. This is a phenomenon deemed coherent in an alternative popular model of belief, that of sets of desirable gambles; a connection which will prove important. It is this difference which means the model of this paper is not equivalent to the credal set model, although it is closely connected. We will show how this enables the model to avoid certain key objections to the credal set model of belief, as given by Walley [68]. In particular, it allows for accommodation of strict preferences and similarly can avoid the probability-zero challenges for accommodating updating.

Walley [68] criticised the credal set model for not being appropriately able to capture strict comparative judgements. He argues that if one gamble might lead to a gain and cannot lead to a loss then you should judge it as desirable, preferring it to the status quo. In the credal set account, one typically reads off a judgement that a gamble is desirable when every probability function in the set assigns it a positive expected value. In our formalism, the agent must hold the credal judgement (either explicitly or by commitment) that it has positive expected value. In the credal set framework, the desideratum that such gambles be desirable can be accommodated only if every possible outcome has positive probability according to every member of the credal set. This is not possible in uncountable sample spaces even when only imposing finite additivity.

It is, however, something that we can impose in the probability filter framework. We are able to require that for every possible outcome you judge it to have positive probability. Since we only impose for coherence that your judgements be closed under finitary consequence, you need not hold the judgement that every possible outcome has positive probability, unless the sample space is finite. If the sample space is uncountable, there is no such probability function, so the judgement that every possible outcome has positive probability is satisfied by no probability functions. This would result in triviality. This allows us to force the agent to hold the judgement that the expected value of this gamble is positive, thus strictly preferring it to the status quo, for every gamble which might lead to a gain and cannot lead to a loss.

This will also allow us to obtain a simple treatment for conditionalisation. This is another criticism by Walley [68] of the credal set model. Conditioning on events of probability zero cause significant difficulties in the theory of probability and similarly in the credal set model. But since we can impose of each possible outcome that you judge it to have positive probability, we can make use of simple Bayesian conditionalisation on events with non-zero probability to characterise rational updating for the model.

To light of these challenges, Walley [68] argues that one should instead work directly with a mathematical model for uncertainty which specifies those gambles that the agent judges as desirable or preferable to the status quo, with no explicit reference to an underlying probability account. This is a prominent and popular alternative model of belief [68, 45, 69, 52]. A key result of the current paper is that unlike credal sets, the model we propose is sufficiently informative to encompass coherent sets of desirable gambles. That is, we can understand a judgement that a gamble is desirable as providing a probability constraint, that you judge it to have positive expected payout. We will show that the usual notion of coherence on sets of desirable gambles matches exactly that of being closed and consistent under finitary probabilistic entailment (along with the assumption that each possible outcome is judged to have positive probability).

This paper proceeds as follows: We introduce the proposed model in Section 2 as a collection of probability constraints and giving the notion of coherence as being a filter on probability functions. We will show how it can accommodate a version of regularity (Section 2.5) which allows it to avoid probability zero difficulties in accommodating updating (Section 2.6).

In Section 3, we discuss the relationship between the model of probability filters and that of sets of desirable gambles. The most important results of this paper are that the probability filter framework encompasses that of coherent sets of desirable gambles given their usual axioms (Theorems 3.7 and 3.10). More precisely, once pointwise regularity is imposed, the usual axioms on coherence of sets of desirable gambles are exactly the ones that are characterised by the probability filter model. Since the probability filter model is simply imposing finite probabilistic entailment this is to show a close connection with these axioms and the laws of probability once only finite consequences are considered.

In Section 4 we will move to discuss the relationship between probability filters and a more general model proposed for uncertainty given by coherent choice functions or coherent sets of desirable gamble sets, as studied in De Cooman, Van Camp and De Bock [12, 13, 61, 16], building on Seidenfeld et al. [53]. We consider a judgement of a gamble *set* as desirable to be a credal judgement and satisfied by the probability functions which give positive expectated value to some member of the set. This is to apply an analogue of E-admissibility rather than Maximality. Our results show that requiring that judgements of this form to be closed and consistent under finitary probabilistic consequence results in the usual axioms of coherence plus a so-called mixing axiom.

De Cooman, De Bock and Van Camp [16] have recently considered using filters of coherent sets of desirable gambles themselves. This offers a more general model, as it encompasses all coherent choice functions rather than just those satisfying the mixing axiom. We offer a suggestion to use probability filters as one's model of belief and accommodate other choice functions by inclusion of a further parameter describing how one structures decisions. This allows for a model of belief which remains close to that of the credal set model, keeping the account within the tradition of capturing the agents belief state with models based on probabilities.

This paper argues that this probability filter model, which is closely connected to the credal set model, can avoid the objections by Walley [68] to the credal set model. Its results also identify the axioms on desirability corresponding to consistency and closure under finitary probabilistic entailment amongst judgements of gambles having positive expected value. Once we additionally include pointwise regularity, which is to include the judgement that each possible outcome has positive probability, then the usual axioms on coherence on desirable gambles are exactly characterised.

2 The probability filters model of belief

2.1 Probability constraints and coherence

In this section we introduce the new mathematical framework for modelling the agent's belief state.

Representations are always relative to an underlying space of possible outcomes, Ω. Formally, Ω is simply a non-empty set. It represents the possible ways that the world might be. If we are just focusing on her opinions about the outcomes of a specific experiment, Ω would consist of the different possible experimental outcomes, where it is guaranteed that exactly one of these outcomes obtains.

Setup 2.1. A (finitely additive) **probability function** is $p : \wp(\Omega) \to \mathbb{R}_{\geq 0}$ which is normalised, where $p(\Omega) = 1$, and finitely additive, where $p(F \cup E) = p(F) + p(E)$ for F, E disjoint subsets of Ω.

Probs is the collection of all probability functions.

A **probability constraint** is any set of probability functions, $C \subseteq \text{Probs}$.

For a probability constraint $C \subseteq \text{Probs}$, $\overline{C}$ is the **complement** of C, relative to Probs, i.e., $\overline{C} = \text{Probs} \setminus D$.

Our proposed model of belief is closely connected to the credal set model. It is supposed that the agent's belief state is characterised by her 'judgements'; both those that she explicitly holds and some that she is committed to. We are building a model closely related to the credal set model, and understand her judgements as satisfied by some probability functions and not by others, thus imposing a probability constraint. Van Fraassen calls these 'epistemic judgements' [64, 63]. We sometimes call them 'credal judgements'.

An agent's judgements may be expressed with sentences such as "rain seems at least as likely as snow to me". This determines a constraint on credence functions, and in particular, imposes a constraint on the (finitely additive) probability functions which satisfy it. We will write an expression like this more concisely as "pr(Rain) $\geq$ pr(Snow)" and use $[\![\ldots]\!]$ to denote the set of probability functions that satisfy it. For example:

$$[\![\text{pr}(\textsc{Rain}) \geq \text{pr}(\textsc{Snow})]\!] = \{p \in \text{Probs} \mid p(\textsc{Rain}) \geq p(\textsc{Snow})\}.$$

The judgement expressed by "it seems twice as likely that John passes the exam than that Billy passes the exam" is written more concisely and determines the probability constraint as follows:

$$\begin{aligned}&[\![\text{pr}(\textsc{JohnPasses}) = 2\,\text{pr}(\textsc{BillyPasses})]\!]\\&\quad = \{p \in \text{Probs} \mid p(\textsc{JohnPasses}) = 2\,p(\textsc{BillyPasses})\}.\end{aligned}$$

In general, we will express probability constraints by using formulations in a language with 'pr' and denote the set of probabilities satisfying them with $[\![\ldots \text{pr} \ldots]\!] = \{p \in \text{Probs} \mid \ldots p \ldots\}$. The form of credal judgements that the agent can make and restrictions to the class of probability constraints at stake will be returned to in Section 2.4. In our main account, we allow for arbitrary subsets of Probs.

We will use these probability constraints to provide our mathematical model of the belief-state, or credal state of the agent. We propose this as a model of indeterminate rather than imprecise probabilities, in the sense of Levi [36, 38]. We propose replacing the role of a credal set with that of a probability filter, which encodes the probability judgements, or credal judgements, of the agent.

We understand such judgements as imposing restrictions on (personal) probability functions or 'credence' functions. In line with Levi [38, 39, 40] we might take them to be restrictions on the permissible probabilities, or perhaps restrictions on those credence functions which the agent deems it legitimate to use when, for example, computing expected utilities. Van Fraassen's [63, 64, 65, 66] view is that one's (personal) probabilities are vague and to be treated along supervaluational lines. The members of one's credal set are the "precisifications" and the judgements are the determinate properties. We propose taking seriously Van Fraassen's view that the judgements, or determinate properties, are primary and our modelling of the agent's belief state works directly with such judgements.

We focus on just the probability functions satisfying the judgements as opposed to other real-valued 'credence' functions which fail to satisfy the axioms of probability. This is legitimate once one stipulates that a coherent agent is committed to the judgement that "pr satisfies the axioms of probability". Or alternatively, simply defines the notion of coherent entailment according to the probability functions satisfying the judgements. This extends the standard view that the members of one's credal set should satisfy the axioms of probability.

It allows us to obtain nice normative properties such as avoidance of susceptibility to a Dutch book.[5] Thinking of the agent's credal judgements as ruling out various credence functions as inadmissible or illegitimate, as identifying certain features of credences she is committed to and others that she rules out, then she should be committed to probabilistic coherence of the credence function. This may perhaps be justified as a consequence of arguments that credences be probabilistic, for example, due to their success at providing estimates of the truth-values [23, 27]. In the determinacy approach to credal sets and credal judgements more generally, then it is plausibly determinate that it is recommended that one's credences be probabilistic, and thus, we propose a rational requirement that it be determinate that credences are probabilistic.[6]

Our proposed mathematical representation of the belief state of our agent is to collect together those probability constraints which the agent is committed to, which we will sometimes gloss as those that she believes. Formally, then, her belief state is represented by a collection of probability constraints $\mathcal{F} \subseteq \wp(\text{Probs})$. This will include the probabilistic content of her explicitly held credal judgements but also those probability constraints that she is committed to by virtue of a finitary notion of probabilistic entailment which we will further discuss in Section 2.3.

Definition 2.2. A set of probability constraints, i.e., a set of subsets of Probs, $\mathcal{F} \subseteq \wp(\text{Probs})$, is **coherent** if it is a **proper filter**, i.e., it satisfies the following axioms:

($\mathrm{F}_\cap$) For any $C, D \subseteq \text{Probs}$, if $C, D \in \mathcal{F}$ then $C \cap D \in \mathcal{F}$.

($\mathrm{F}_\supseteq$) For any $C, D \subseteq \text{Probs}$, if $C \in \mathcal{F}$ and $D \supseteq C$ then $D \in \mathcal{F}$.

($\mathrm{F}_{\neq\varnothing}$) $\mathcal{F} \neq \varnothing$.

($\mathrm{F}_{\text{Proper}}$) $\varnothing \notin \mathcal{F}$.

[5] This is the defence of the assumption offered by Joyce [28, p292], although see Section 3.1.1.

[6] Under the proposal of Campbell-Moore [5, §4.5]. Various arguments against accuracy for imprecise credences try to determine the accuracy of the indeterminate credal state [54, 43, 51, 8]; see also Konek [33]. The line of thought here is instead just measuring accuracy of the *members* of her credal state to determine overall constraints rather than a measure of the accuracy of the indeterminate credal state.

The notion of being a proper filter is standard from topology or propositional logic [22, 10]. Since we are representing uncertainty with a filter on the probabilities, we will call these **probability filters**.

Axioms ($\mathrm{F}_{\neq\varnothing}$) and ($\mathrm{F}_{\mathrm{Proper}}$) ensure that a coherent $\mathcal{F}$ is non-trivial. Axiom ($\mathrm{F}_{\neq\varnothing}$) says that there has to be some probability constraint that she is committed to, if only Probs, the set of all probability functions, itself. Axiom ($\mathrm{F}_{\mathrm{Proper}}$) says that she does not believe $\varnothing$. The substantive axioms are axioms ($\mathrm{F}_{\cap}$) and ($\mathrm{F}_{\supseteq}$) saying that her believed probability constraints should be closed under finite intersections and supersets. This is to say that we have closed her judgements under finite consequences, which, when including our axiom ($\mathrm{F}_{\neq\varnothing}$) and the assumption that it is a filter on Probs, i.e., ensuring that $\mathrm{Probs} \in \mathcal{F}$, is to say that we have closed her judgements off under a finite notion of probabilistic consequence.

The axioms on coherence ensure that we have the following result.

Proposition 2.3 (No Confusion)**.** [7] *If $\mathcal{F}$ is coherent, then there is no $D \subseteq \mathrm{Probs}$ where $D \in \mathcal{F}$ and $\overline{D} \in \mathcal{F}$.*

Proof. If both $D \in \mathcal{F}$ and $\overline{D} \in \mathcal{F}$, then by axiom ($\mathrm{F}_{\cap}$), we also have $D \cap \overline{D} = \varnothing \in \mathcal{F}$, contradicting axiom ($\mathrm{F}_{\mathrm{Proper}}$). □

This model allows for indeterminacy and a mode of suspension as there may be probability constraints with neither $D \in \mathcal{F}$ nor $\overline{D} \in \mathcal{F}$.

We say that $\mathcal{F}'$ is **at least as committal** as $\mathcal{F}$ when $\mathcal{F} \subseteq \mathcal{F}'$. If she suspends on a probability constraint D, she may become more committal by adopting it or by adopting its complement. This will be formally shown in Proposition 2.12.

2.2 Examples of probability filters

In this section, we give some important examples of coherent sets of probability constraints, or probability filters, $\mathcal{F}$.

Proposition 2.4. *For any non-empty set of probabilities, $\varnothing \neq C^* \subseteq \mathrm{Probs}$, we can associate a set of probability constraints, $\mathcal{F}_{C^*} \subseteq \wp(\mathrm{Probs})$, defined by:*

$$\mathcal{F}_{C^*} := \{D \subseteq \mathrm{Probs} \mid D \supseteq C^*\}. \tag{1}$$

Then:

(i) *$\mathcal{F}_{C^*}$ is coherent.*

(ii) *$\mathcal{F}_{C^*}$ is the least committal coherent probability filter where the probability constraint C^* is believed. That is, for any coherent $\mathcal{F}'$ with $C^* \in \mathcal{F}'$, we have $\mathcal{F}' \supseteq \mathcal{F}_{C^*}$.*

(iii) *For $C^* \neq C$, $\mathcal{F}_{C^*} \neq \mathcal{F}_C$.*

Proof. It is immediate to check that $\mathcal{F}_{C^*}$ satisfies all the axioms for coherence, as given in Definition 2.2.

To observe that $\mathcal{F}_{C^*}$ is the least committal coherent opinion set where C^* is believed, note that any coherent $\mathcal{F}'$ with $C^* \in \mathcal{F}'$ also has any superset of C^*, any $D \supseteq C^*$, also in $\mathcal{F}'$, by axiom ($\mathrm{F}_{\supseteq}$). And thus $\mathcal{F}' \supseteq \mathcal{F}_{C^*}$.

If $C^* \neq C$, then either $C^* \not\supseteq C$ or $C \not\supseteq C^*$, so it is trivial to observe that $\mathcal{F}_{C^*} \neq \mathcal{F}_C$. □

[7] The name follows Quaeghebeur et al. [46, p.73]

This result shows us that the credal set model of belief is encompassed by the probability filter model. In the credal set model coherent belief states are given by a set of probability functions, C^*. She is said to have a credal judgement when every probability function in C^* satisfies it. This is just to say that she believes the probability constraints which are supersets of her set of probability functions. This is what is given by $\mathcal{F}_{C^*}$.

Filters that have the form $\mathcal{F}_{C^*}$ for some $C^* \subseteq \text{Probs}$ are, as standard for filters, called **principal filters**.[8] Not all coherent sets of probability constraints have this form. There are non-principal filters, and thus coherent sets of probability constraints that do not correspond to any credal set.

Proposition 2.5. *There are coherent $\mathcal{F}$ where $\mathcal{F}$ does not have the form $\mathcal{F}_{C^*}$ for any $C^* \subseteq \text{Probs}$. (Where $\mathcal{F}_{C^*}$ is specified as in Eq. (1)).*

Moreover, there are coherent $\mathcal{F}$ where $\bigcap \mathcal{F} = \varnothing$; that is, there is no $p^* \in \text{Probs}$ which is a member of every $C \in \mathcal{F}$.

We couch the proof of this proposition as an example. It is a probability filter that we will return to throughout the paper.

Example 2.6. Fix $\Omega = \{\text{H}, \text{T}\}$, representing the outcome of a coin toss. We will describe a probability filter $\mathcal{F}_{\text{InfBiased}}$, which does not correspond to any single set of probabilities.

$$\mathcal{F}_{\text{InfBiased}} := \left\{ C \subseteq \text{Probs} \;\middle|\; \begin{array}{l} \text{there is some } \epsilon \in \mathbb{R} \text{ with } \epsilon > 0 \text{ and} \\ C \supseteq [\![0.5 < \text{pr(H)} < 0.5 + \epsilon]\!] \end{array} \right\}.$$

Recall our notation $[\![0.5 < \text{pr(H)} < 0.5+\epsilon]\!] = \{p \in \text{Probs} \,|\, 0.5 < p(\text{H}) < 0.5+\epsilon\}$.

To observe that this satisfies axiom ($\text{F}_\cap$), consider $C_1 \supseteq [\![0.5 < \text{pr(H)} < 0.5+\epsilon_1]\!]$ and $C_2 \supseteq [\![0.5 < \text{pr(H)} < 0.5+\epsilon_2]\!]$. Then observe that $C_1 \cap C_2 \supseteq [\![0.5 < \text{pr(H)} < 0.5 + \min\{\epsilon_1, \epsilon_2\}]\!]$, so $C_1 \cap C_2 \in \mathcal{F}_{\text{InfBiased}}$. The remaining axioms are easy to check.

For any $p \in [\![\text{pr}(H) > 0.5]\!]$, that is, $p \in \text{Probs}$ with $p(\text{H}) > 0.5$, there is some $\epsilon^* > 0$ such that $p(\text{H}) \geq 0.5 + \epsilon^*$ (just take $\epsilon^* = p(\text{H}) - 0.5$), so $p \notin [\![\text{pr}(H) < 0.5 + \epsilon^*]\!]$. This shows that there is no p such that $p \in C$ for all $C \in \mathcal{F}_{\text{InfBiased}}$, i.e., $\bigcap \mathcal{F}_{\text{InfBiased}} = \varnothing$.

This represents an agent for whom it seems that the coin is more likely to land heads than tails but not by any particular amount. She holds the judgement $[\![\text{pr(H)} > 0.5]\!]$ and also the infinitely many judgements $[\![\text{pr(H)} < 0.5 + \epsilon]\!]$ for every ϵ.

Such an agent judges that an even-odds bet towards heads has positive expected value, but for any price to pay for entering the exchange, she judges the trade to have a negative expected value.

2.3 Natural Extension

If we know some of the agent's believed probability constraints, we can determine additional ones that she is committed to by virtue of coherence.

[8] In fact, principal filters are exactly those that are also closed under *infinite* intersections. We thus have an equivalence between the credal set framework and probability filters which satisfy this infinite intersection property. They can be seen as collecting those probability constraints which are infinitary probabilistic consequences of her explicitly held credal judgements.

Proposition 2.7. *For coherent $\mathcal{F}$, if $D_1, \ldots, D_n \in \mathcal{F}$ and $C \supseteq D_1 \cap \ldots \cap D_n$ then also $C \in \mathcal{F}$.*

Proof. Suppose $D_1, \ldots, D_n \in \mathcal{F}$. By iterated uses of axiom ($F_\cap$), we can see that $D_1 \cap \ldots \cap D_n \in \mathcal{F}$. Then by axiom ($F_\supseteq$), $C \in \mathcal{F}$. □

We give two illustrative instances of Proposition 2.7

Example Fact 2.8. *For coherent $\mathcal{F}$, if $[\![\mathrm{pr}(E) > 0.6]\!] \in \mathcal{F}$ then $[\![\mathrm{pr}(E) > 0.2]\!] \in \mathcal{F}$.*

Proof. This follows from axiom ($F_\supseteq$), since

$$[\![\mathrm{pr}(E) > 0.6]\!] \subseteq [\![\mathrm{pr}(E) > 0.2]\!].$$ □

It can also be seen as a special case of Proposition 2.7 with $n = 1$.

Example Fact 2.9. *For coherent $\mathcal{F}$, if $[\![\mathrm{pr}(E) \leq 0.3]\!] \in \mathcal{F}$ and $[\![\mathrm{pr}(F) \leq \mathrm{pr}(E)]\!] \in \mathcal{F}$ then $[\![\mathrm{pr}(E \cup F) \leq 0.6]\!] \in \mathcal{F}$.*

Proof. For $p \in \mathrm{Probs}$, if $p(E) \leq 0.3$ and $p(F) \leq p(E)$ then also $p(F) \leq 0.3$ and therefore, by the probability axioms, $p(E \cup F) \leq 0.3 + 0.3 = 0.6$. I.e.,

$$[\![\mathrm{pr}(E) \leq 0.3]\!] \cap [\![\mathrm{pr}(F) \leq \mathrm{pr}(E)]\!] \subseteq [\![\mathrm{pr}(E \cup F) \leq 0.6]\!]$$

So, this immediately follows from Proposition 2.7. □

In fact Proposition 2.7 fully characterises coherence, and determines the so-called natural extension: the least committal probability filter containing a given collection of probability constraints.

Definition 2.10. For non-empty $\mathcal{E} \subseteq \wp(\mathrm{Probs})$, $\mathrm{ext}(\mathcal{E})$ is defined by: $C \in \mathrm{ext}(\mathcal{E})$ iff there are some finitely many members of $\mathcal{E}$, $D_1, \ldots, D_n$, such that $C \supseteq D_1 \cap \ldots \cap D_n$. That is:

$$\mathrm{ext}(\mathcal{E}) := \left\{ C \subseteq \mathrm{Probs} \;\middle|\; \begin{array}{l} \text{there are some } D_1, \ldots, D_n \in \mathcal{E} \\ \quad (\text{where } n \in \mathbb{N} \setminus \{0\}), \\ \text{with } C \supseteq D_1 \cap \ldots \cap D_n. \end{array} \right\}.$$

In our next proposition, we give a standard result regarding filters [see, e.g., 1, p.58].

The first part gives a useful criterion for when a collection of judgements can be coherently extended to a filter: it is when the probability constraints are finitely consistent. This means that for any finitely many credal judgements, there is at least one probability function which satisfies them all.

The second part shows that under such conditions, ext is indeed the usual notion of natural extension, giving the least committal coherent extension.

Proposition 2.11. *Suppose $\mathcal{E} \subseteq \wp(\mathrm{Probs})$ is non-empty. Then:*

- *There is a coherent $\mathcal{F} \supseteq \mathcal{E}$ iff $\varnothing \notin \mathrm{ext}(\mathcal{E})$; i.e., iff for any $D_1, \ldots, D_n \in \mathcal{E}$, $D_1 \cap \ldots \cap D_n \neq \varnothing$ (this is usually called the 'finite intersection property').*
- *If $\varnothing \notin \mathrm{ext}(\mathcal{E})$, then $\mathrm{ext}(\mathcal{E})$ is coherent and is the least committal coherent $\mathcal{F} \supseteq \mathcal{E}$.*

We include the proof because it is important in the results that follow.

Proof. As we noted in Proposition 2.7, if $D_1, \ldots, D_n \in \mathcal{F}$ with $C \supseteq D_1 \cap \ldots \cap D_n$ then $C \in \mathcal{F}$. So, for any coherent $\mathcal{F} \supseteq \mathcal{E}$, we must have $\mathcal{F} \supseteq \text{ext}(\mathcal{E})$.

So it suffices to check that $\text{ext}(\mathcal{E})$ is coherent iff $\varnothing \notin \text{ext}(\mathcal{E})$. Axioms $(F_\cap)$ and $(F_\supseteq)$ hold for any $\text{ext}(\mathcal{E})$ because we have forced $\text{ext}(\mathcal{E})$ to be closed under intersection and supersets by definition. (For the supersets, use $n = 1$.) It is non-empty by assumption on $\mathcal{E}$, since any $C \in \mathcal{E}$ is also in $\text{ext}(\mathcal{E})$. So the only remaining axiom for coherence is (F_{Proper}). Thus, $\text{ext}(\mathcal{E})$ is coherent iff (F_{Proper}) holds of it, i.e., iff $\varnothing \notin \text{ext}(\mathcal{E})$. □

Another useful case is when $\mathcal{E}$ is already closed under finite intersections. In this case, $\text{ext}(\mathcal{E})$ just takes supersets; that is $C \in \text{ext}(\mathcal{E})$ iff $C \supseteq D$ for some $D \in \mathcal{E}$.

This natural extension can be seen as the mechanism for taking the agent's explicitly held credal judgements and determining additional ones she is committed to. This allows her belief state to be stored in the form of a few explicit judgements. We can also directly present a version of this applied directly to credal judgements rather than going via their associated probability constraints. A collection of judgements is said to **probabilistically entail** a further judgement if every probability function satisfying the former judgements also satisfies the latter. They are said to **finitarily probabilistically entail** another judgement if a finite subset of them probabilistically entails that additional judgement. Closing a set under finite probabilistic entailment is exactly what is done by the natural extension notion given here. Coherent agents are said to be committed to additional judgements which are finitarily probabilistically entailed by her held judgements. This is a finitary version of the approach of Van Fraassen [64, p. 252].

Finally, we present the following result, which shows an important aspect of the model of belief showing that it properly accommodates suspension of judgement. If $C \notin \mathcal{F}$ and $\overline{C} \notin \mathcal{F}$, then $\mathcal{F}$ could become more committal either way.

Proposition 2.12. *Suppose $\mathcal{F}$ is coherent. If $\mathcal{F}$ suspends judgement on a probability constraint $D^* \subseteq \text{Probs}$, i.e., $D^* \notin \mathcal{F}$ and $\overline{D^*} \notin \mathcal{F}$, then there is a coherent probability filter $\mathcal{F}_{\text{bel}}$ which is at least as committal as $\mathcal{F}$, i.e., $\mathcal{F}_{\text{bel}} \supseteq \mathcal{F}$, and which believes D^*, i.e., $D^* \in \mathcal{F}_{\text{bel}}$; and there is another coherent probability filter $\mathcal{F}_{\text{disbel}}$ which is at least as committal as $\mathcal{F}$, i.e., $\mathcal{F}_{\text{disbel}} \supseteq \mathcal{F}$, and which disbelieves D^*, i.e., $\overline{D^*} \in \mathcal{F}_{\text{disbel}}$*

Proof. Put $\mathcal{F}_{\text{bel}} := \text{ext}(\mathcal{F} \cup \{D^*\})$. We just need to check that $\mathcal{F}_{\text{bel}}$ is coherent. By Proposition 2.11, it suffices to check that for any $C_1, \ldots, C_n \in \mathcal{F}$ we have that $D^* \cap C_1 \cap \ldots \cap C_n \neq \varnothing$. If $D^* \cap C_1 \cap \ldots \cap C_n = \varnothing$ then $\overline{D^*} \supseteq C_1 \cap \ldots \cap C_n$; and thus by Proposition 2.7, already $\overline{D^*} \in \mathcal{F}$ by coherence of $\mathcal{F}$. So, since we have assumed that $\overline{D^*} \notin \mathcal{F}$, we know that $\mathcal{F}_{\text{bel}}$ is coherent. By construction, it is more committal than $\mathcal{F}$ and contains D^*.

Similarly, we can see that $\mathcal{F}_{\text{disbel}} := \text{ext}(\mathcal{F} \cup \{\overline{D^*}\})$ is coherent since $D^* \notin \mathcal{F}$. □

As a consequence of this, the maximal coherent $\mathcal{F}$ are those where for every $C \subseteq \text{Probs}$ either $C \in \mathcal{F}$ or $\overline{C} \in \mathcal{F}$. These are typically called ultrafilters. The principal ultrafilters correspond to precise probabilities. The non-principal, or free, ultrafilters are closely related to hyperreal valued probability functions.

We can also see as a consequence of this that this framework provides a strong belief structure in the sense of De Cooman [14], that is, that any coherent filter is the infimum of its dominating maximal belief models, which are the ultrafilters.

2.4 Restricting the range of credal judgements

Our mathematical model of the agent's belief state has moved from discussion of credal judgements to directly working with probability constraints. Our notion of coherence just considers the probability functions satisfying the judgements so the constraint on probabilities imposed by the judgement is all that matters. In our general account, we have allowed arbitrary subsets of Probs to be probability constraints and a candidate for inclusion in $\mathcal{F}$. In doing this, we have ensured that we do not restrict to limited forms of judgements, such as that of merely comparative probabilities or upper and lower probabilities. We have also, for example, allowed the non-convex probability constraint, $\{p \in \text{Probs} \mid p(E) = 0.2 \text{ or } p(E) = 0.3\}$, as something that can be in $\mathcal{F}$. Such non-convex sets will almost always be included in coherent $\mathcal{F}$, if only because they are supersets of members of $\mathcal{F}$.

The formalism that we have developed, is compatible with restrictions to a range of "representationally significant" probability constraints. These could, for example, be convex sets of probability functions, or judgements certain gambles as having positive expected value. Whilst it is possible to restrict the framework in such ways, one needs to be careful to consider whether important differences of opinion are being ruled out.

There are various proponents of restricting to convex credal sets [e.g., 37, 19, 68] but also those who object to it [34, 35, 28, 53, 63], and in much of philosophy community it is not assumed [32, 2, 50]. The model of credal sets commonly adopted by authors who reject convexity is unconstrained sets of probabilities [34, 35, 28, 32, 2, 50].

Moreover, at least if the decision rule of E-admissibility is used, Seidenfeld et al. [53] shows that arbitrary sets of probabilities can lead to different choice functions, giving support to the claim that arbitrary constraints on probabilities are to be considered. This leads to our suggestion of working with arbitrary filters on Probs which may be characterised by arbitrary sets of probabilities.[9]

Suppose we have some specified range of judgements which are to be considered, $\mathcal{R} \subseteq \wp(\text{Probs})$. How should we implement this restriction into the framework?

The approach in line with De Cooman et al. [16] would be to restrict to filters not defined over $\wp(\text{Probs})$ but only over $\mathcal{R}$. If $\mathcal{R}$ is closed under finite intersections, the analogous axioms from Definition 2.2 can be given. This is the case for the restriction proposed for [16].

An alternative approach to implement a restriction is to still allow for arbitrary filters as mathematical tools, but say that only certain aspects of it are relevant for capturing the agents belief state. This is the approach suggested in Joyce [28],

[9]In Section 4.7 we show a case where a seemingly very general class of judgements still rules out certain seemingly important differences of opinion. It shows that distinct probability filters may nonetheless result in the same choice function.

although his point is to reject various such restrictions. Using his terminology, we can say that $\mathcal{F}$ and $\mathcal{F}'$ "encode the same beliefs about Ω" when $\mathcal{R} \cap \mathcal{F} = \mathcal{R} \cap \mathcal{F}'$.

Given any filter $\mathcal{F}'$ we could then identify $\mathsf{cl}_{\mathcal{R}}(\mathcal{F}') = \mathrm{ext}(\mathcal{F}' \cap \mathcal{R})$, the least committal coherent set of probability constraints which encodes the same beliefs as $\mathcal{F}$.

Filters which can be generated in such a way are those that satisfy a further axiom:

($\mathrm{F}_{\mathrm{Restr}(\mathcal{R})}$) If $C \in \mathcal{F}$ then there are finitely many $R_1, \ldots, R_n \in \mathcal{F} \cap \mathcal{R}$ ($n \in \mathbb{N} \setminus \{0\}$) such that $C \supseteq R_1 \cap \ldots \cap R_n$.

This axiom is satisfied just when the agent's belief state is characterised by credal judgements that have the form accommodated in $\mathcal{R}$.

For example, if $\mathcal{R} = \{C \subseteq \mathrm{Probs} \mid C \text{ is convex}\}$ then we can still have non-convex probability constraints in $\mathcal{F}$, we are not restricting $\mathcal{F}$ to only be defined on convex sets. But non-convex constraints are only in $\mathcal{F}$ when they are merely implications of convex constraints. For example if $[\![\mathrm{pr}(H) = 0.2]\!] \in \mathcal{F}$ then also $[\![\mathrm{pr}(H) = 0.2 \text{ or } \mathrm{pr}(H) = 0.3]\!] \in \mathcal{F}$.

We will continue to work in the general mathematical framework of arbitrary filters of probabilities, as the most natural extension of the model of arbitrary credal sets. However, interest in the model is not dependent on this fact and it can be restricted to various classes of "representationally significant" probability constraints in one of these ways. In Sections 3.6 and 4.7, we will discuss some possible restrictions, using axiom ($\mathrm{F}_{\mathrm{Restr}(\mathcal{R})}$) as our general way of restricting the framework.

2.5 Regularity

One can impose a further axiom on probability filters which is closely related to regularity.

A credal set is said to satisfy credal regularity when every probability function in the set assigns non-zero probability to each possible outcome. Regularity for credences has been defended by [29, 56, 58, 41, 57].

Credal regularity results in the attractive feature that if a gamble has some possibility of gain and no possibility of loss, then every probability function in your credal set evaluates it to have positive expected gain, so you judge it as preferable to the status quo. Walley [68] argues that this is essential for accommodating strict preferences.

Whilst it is a plausible principle with attractive consequences, this principle is impossible to satisfy when there are uncountably many possible outcomes. There are no finitely additive probability functions which assign positive probability to uncountably many possible outcomes. There are thus no regular probability functions and thus no credal sets satisfying credal regularity. Lewis [41], Skyrms [57] advocate moving to hyperreal probability functions to maintain regularity. In the probability filter framework, however, we do not need to introduce hyperreals explicitly, instead the flexible structure of one's judgements can accommodate this.

We can require of each possible outcome that you judge it to have positive probability. That is, for each $\omega \in \Omega$, $[\![\mathrm{pr}(\omega) > 0]\!] \in \mathcal{F}$. We will call this *pointwise regularity.*

Definition 2.13. $\mathcal{F}$ is **pointwise regular** iff

($\mathrm{F}_{\text{reg-pw}}$) For all $\omega \in \Omega$, $[\![\mathrm{pr}(\omega) > 0]\!] \in \mathcal{F}$.

This is "pointwise" in the sense that it only imposes of each $\omega \in \Omega$ individually that $[\![\mathrm{pr}(\omega) > 0]\!] \in \mathcal{F}$. This will entail that $[\![\mathrm{pr}(\omega_1) > 0 \text{ and } \ldots \text{ and } \mathrm{pr}(\omega_n) > 0]\!]$ for any finitely many $\omega_1, \ldots, \omega_n \in \Omega$. It will not, however, entail that $[\![\mathrm{pr}(\omega) > 0 \text{ for all } \omega \in \Omega]\!] \in \mathcal{F}$ unless Ω is finite.

When Ω is uncountable, $[\![\mathrm{pr}(\omega) > 0 \text{ for all } \omega \in \Omega]\!]$ is just $\varnothing$: there are no such probability functions. Thus, by axiom ($\mathrm{F}_{\mathrm{Proper}}$), it is not in $\mathcal{F}$. This is not, however, enough to show that pointwise regularity cannot be imposed, as it does not follow from pointwise regularity because of our restriction to finite entailment.

In fact, for any space of possible outcomes, we can find coherent sets of probability constraints which are pointwise regular.

Proposition 2.14. *For any Ω, there is a coherent $\mathcal{F}$ which is pointwise regular.*

This can be seen as a consequence of our forthcoming Theorem 3.10, but it can also be proved directly.

Proof. Consider $\mathrm{ext}(\{[\![\mathrm{pr}(\omega) > 0]\!] | \omega \in \Omega\})$. This satisfies axiom ($\mathrm{F}_{\text{reg-pw}}$). Due to Proposition 2.11, we just need to show that for any finitely many $\omega_1, \ldots, \omega_n \in \Omega$, $[\![\mathrm{pr}(\omega_1) > 0]\!] \cap \ldots \cap [\![\mathrm{pr}(\omega_n) > 0]\!] \neq \varnothing$.

Fix some finitely many $\omega_1^*, \ldots, \omega_n^*$. Define a probability function p^* which is uniform over these finitely many worlds. This is given by $p^*(E) = \frac{\#(E \cap \{\omega_1^*, \ldots, \omega_n^*\})}{n}$, with # denoting the cardinality. As $p^*(\omega_i^*) > 0$ for each i, in fact, $p^*(\omega_i^*) = {}^1\!/\!{}_n$, then $p^* \in [\![\mathrm{pr}(\omega_i^*) > 0]\!]$ for each $i = 1, \ldots, n$. Thus $[\![\mathrm{pr}(\omega_1) > 0]\!] \cap \ldots \cap [\![\mathrm{pr}(\omega_n) > 0]\!] \neq \varnothing$, as required. □

Proposition 2.15. *Suppose $\mathcal{F}$ is coherent and pointwise regular. If $E \subseteq \Omega$ is such that $E \neq \varnothing$ then $[\![\mathrm{pr}(E) > 0]\!] \in \mathcal{F}$.*

Proof. Consider any $\omega^* \in E$. By pointwise regularity, $[\![\mathrm{pr}(\omega^*) > 0]\!] \in \mathcal{F}$. For any probability function, p, $p(\omega^*) > 0$ implies $p(E) > 0$, so $[\![\mathrm{pr}(E) > 0]\!] \supseteq [\![\mathrm{pr}(\omega^*) > 0]\!]$. Thus, by axiom ($\mathrm{F}_{\supseteq}$), $[\![\mathrm{pr}(E) > 0]\!] \in \mathcal{F}$. □

2.6 Updating and Conditionalisation

By being able to impose pointwise regularity, we can avoid difficulties faced in accommodating updating for the probability framework. This was a key criticism of the credal set model by Walley [68] and a proposed significant benefit of our probability filter model.

Upon learning a proposition $E \subseteq \Omega$ what updated belief state should you adopt? There is a simple answer for the standard probability framework at least when $p(E) > 0$, that is, that you should apply the ratio formula or Bayesian conditionalisation. That is, adopt an updated probability $p(\cdot \mid E)$ given by $p(F \mid E) = \frac{p(F \cap E)}{p(E)}$ for $F \subseteq \Omega$. Difficulties arise only when $p(E) = 0$.

This is typically applied to credal set characterisation of imprecise probability in a pointwise manner [e.g. 28, 3I]: update each member of one's credal set, C^*, by standard Bayesian conditionalisation, at least when $p(E) > 0$ for every

$p \in C^*$, i.e., $C^* \subseteq [\![\mathrm{pr}(E) > 0]\!]$. That is, when $C^* \subseteq [\![\mathrm{pr}(E) > 0]\!]$, then after learning E you should adopt $C^*|E := \{p(\cdot \mid E) \mid p \in C^*\}$.

We can extend this pointwise approach to the probability filter framework in a natural way: If your pre-learning filter believed probability constraint C with $C \subseteq [\![\mathrm{pr}(E) > 0]\!]$, then after learning E, you should believe $C|E := \{p(\cdot \mid E) \mid p \in C\}$.

We will extend this idea to $C \not\subseteq [\![\mathrm{pr}(E) > 0]\!]$ by just updating by conditionalisation those members of C which are in $[\![\mathrm{pr}(E) > 0]\!]$, i.e., where $p(E) > 0$.

Definition 2.16.

$$\mathcal{F}|E = \left\{ D \subseteq \mathrm{Probs} \;\middle|\; \begin{array}{l} \text{there is } C \in \mathcal{F} \\ \text{with } D \supseteq \{p(\cdot|E) \mid p \in C \text{ and } p(E) > 0\} \end{array} \right\}$$

We will provide a minimal result showing that this update is always sensible in that the updated probability filter is coherent if the original one was coherent and pointwise regular. Whilst the updated filter will not be pointwise regular as it will contain $[\![\mathrm{pr}(\omega) = 0]\!]$ for $\omega \notin E$, it is pointwise regular *on* E. It is also essentially defined by probability functions which are supported on E. One might instead conceive of $\mathcal{F}|E$ as a filter whose 'points' are probability functions on E rather than on Ω, however we do not pursue this for simplicity with the standard probability update formalism.

Proposition 2.17. *If $\mathcal{F}$ is coherent and pointwise regular, then $\mathcal{F}|E$ is coherent and $[\![\mathrm{pr}(\omega) > 0]\!] \in \mathcal{F}|E$ for all $\omega \in E$. Furthermore, $[\![\mathrm{pr}(E) = 1]\!] \in \mathcal{F}|E$.*

Proof. Since $\mathcal{F} \neq \varnothing$ also $\mathcal{F}|E \neq \varnothing$. It is closed under supersets by definition. It is closed under finite intersections as $\mathcal{F}$ is. To show it is coherent, we thus just need to show that it does not contain $\varnothing$. If $\varnothing \in \mathcal{F}$ then there must be some $C \in \mathcal{F}$ with $C \cap [\![\mathrm{pr}(E) > 0]\!] = \varnothing$. But since $[\![\mathrm{pr}(E) > 0]\!] \in \mathcal{F}$ by pointwise regularity, $C \cap [\![\mathrm{pr}(E) > 0]\!] \in \mathcal{F}$, so $C \cap [\![\mathrm{pr}(E) > 0]\!] = \varnothing$ would contradict the coherence of $\mathcal{F}$.

Consider $\omega^* \in E$. By pointwise regularity of $\mathcal{F}$, $[\![\mathrm{pr}(\omega^*) > 0]\!] \in \mathcal{F}$. And for any $p \in [\![\mathrm{pr}(\omega^*) > 0]\!]$, also $p(E) > 0$ and $p(\omega^* \mid E) > 0$. Thus $[\![\mathrm{pr}(\omega^*) > 0]\!] \supseteq \{p(\cdot|E) \mid p \in [\![\mathrm{pr}(\omega^*) > 0]\!] \text{ and } p(E) > 0\}$, giving us that $[\![\mathrm{pr}(\omega^*) > 0]\!] \in \mathcal{F}|E$.

It remains to show that $[\![\mathrm{pr}(E) = 1]\!] \in \mathcal{F}|E$. Note that $[\![\mathrm{pr}(E) > 0]\!] \in \mathcal{F}$ so $\{p(\cdot|E) \mid p \in [\![\mathrm{pr}(E) > 0]\!]\} \in \mathcal{F}|E$ and that $p(E) = 1$ for every p in this set. Thus $[\![\mathrm{pr}(E) = 1]\!] \in \mathcal{F}|E$. □

Further investigation would provide axioms characterising coherence of a notion of conditional filters, when they can be determined from a single underlying filter in accordance with this update, see also Williams [69].

In Section 3.7, we will show a close relationship with our updated defined here and judgements of called-off gambles.

3 Probability filters and desirable gambles

One of the most prominent models of belief in the imprecise probability literature is to model one's belief by a set of desirable gambles [68, 67, 45, 9, 52, 69]. It is a model that encompasses many other models, such as comparative previsions.

In this section we will compare the probability filter model of belief to the set of desirable gambles model.

3.1 From Probabilities to Estimated Values and Previsions

Estimation of various quantities of interest is important both epistemically and for linking to decision making. We naturally express judgements directly in terms of estimation or expectation. For example, "I expect my grade to increase" might naturally be thought to express "my expectation value of my current grade is lower than my expectation value of my future grade" (Van Fraassen [66, p484-485]). "My expected value of this gamble is positive".

Since finitely additive probability functions determine unique expected values for bounded random variables, such judgements can be understood as constraints on probability functions. For example, a judgement that my grade will increase gives the probability constraint:

$$\begin{aligned}&[\![\mathrm{Exp}_{\mathrm{pr}}(\textsc{Grade}_{\textsc{Future}}) > \mathrm{Exp}_{\mathrm{pr}}(\textsc{Grade}_{\textsc{Current}})]\!]\\ &= \{p \in \mathrm{Probs} \mid \mathrm{Exp}_p(\textsc{Grade}_{\textsc{Future}}) > \mathrm{Exp}_p(\textsc{Grade}_{\textsc{Current}})\}.\end{aligned}$$

A judgement that gamble g has positive expected value can be understood as:

$$[\![\mathrm{Exp}_{\mathrm{pr}}(g) > 0]\!] = \{p \in \mathrm{Probs} \mid \mathrm{Exp}_p(g) > 0\}.$$

We might alternatively replace our use of probabilities in our probability filters directly with (real-valued) estimated value functions and encode not constraints on probability functions but constraints on estimation functions. That is, we can think of the agent as having credal judgements which are satisfied by expected value functions. An expression of an credal judgement of the form "I expect my grade to increase" is then to be understood as satisfied by a collection of expectation functions, those which assign a higher expectation value to my future grade than to my current grade. When she judges that one gamble is better than another, this is to be understood as a credal judgement that her estimated value of the one is higher than the other.

To match the restriction to finitely additive probability functions we would restrict to estimation functions which are monotone, normalised and linear. We will hold her committed to any further judgements which "follow from" her explicitly held judgements, in the sense that any monotone, normalised and linear estimation function satisfying her explicitly held judgements also satisfies the further judgement.

In this paper we are focusing on the relationship to the desirable gambles model of belief and so are interested in particular in her estimates of the payout of gambles. A gamble is a bounded random variable which is interpreted as an uncertain reward, giving a payoff in each state of the world $\omega \in \Omega$, described in a single, predetermined, linear utility scale [45].

Setup 3.1. A **gamble** is a bounded function from Ω to $\mathbb{R}$. $\mathcal{G}$ is the collection of all gambles.

0 is a gamble that takes value 0 at every world.

For gambles f and g, when $f(\omega) \geq g(\omega)$ for all $\omega \in \Omega$, we will say $f \geq g$.

$\mathcal{G}_{\geq 0}$ is the set of gambles where $f \geq 0$ and $\mathcal{G}_{\leq 0}$ is the set of gambles $f \leq 0$.

$\mathcal{G}_{\gneqq 0}$ is the set of gambles where $f \geq 0$ and $f \neq 0$.[10]

$\mathbf{1}_E$ is the indicator gamble for $E \subseteq \Omega$, given by $\mathbf{1}_E := \begin{cases} 1 & \omega \in E \\ 0 & \omega \notin E \end{cases}$.

[10] These are usually simply denoted with $\mathcal{G}_{>0}$, but I keep the $\gneqq$ to highlight the *weak* dominance component, rather than that $g(\omega) > 0$ for all $\omega \in \Omega$.

In this case, her estimated values are typically called her prevision and the restriction to finitely additive probabilities parallels a restriction to so-called linear previsions.

Definition 3.2. $P : \mathcal{G} \to \mathbb{R}$ is a **linear prevision** if it is monotone, normalised and linear. That is:

($\mathrm{P_{Mon}}$) If $g \leq f$ then $P(g) \leq P(f)$.

($\mathrm{P_{Norm}}$) $P(\mathbf{1}_\Omega) = 1$.

($\mathrm{P_{Lin}}$) $P(\alpha g + \beta f) = \alpha P(g) + \beta P(f)$ for $\alpha, \beta \in \mathbb{R}$, $g, f \in \mathcal{G}$.

Prevs is the set of all linear previsions.

We use $P(\omega) := P(\mathbf{1}_{\{\omega\}})$.

Probability functions can be determined from a linear prevision by consideration of the prevision of the indicator gambles. We can also consider the expectation function generated by a finitely additive probability, Exp_p [18, §III] and check that it is a linear prevision in that it is monotone, normalised and linear. In fact, there is an isomorphism between finitely additive probability functions and linear previsions [45]. This isomorphism relies on our restriction to bounded random variables.[11]

It thus makes no formal difference whether we are considering filters over finitely additive probability functions or linear previsions. The advantage of having introduced our model with probabilities is its close link to the credal set model and to highlight the probabilistic foundations of the model. However, since this paper will be focused on the link to the desirable gambles models of belief, there is a simplicity in considering linear previsions instead.

We thus, for the remainder of the paper, move to directly talking about linear previsions and understand credal judgements to impose constraints on linear previsions. The agent is thus represented by a collection of judgements, each ruling out some linear previsions.[12]

We will use concise notation paralleling that introduced for probabilities, such as:

$$[\![\Pr(g) > 0]\!] = \{P \in \mathrm{Prevs} \mid P(g) > 0\}$$

A reader who wants to think about probabilities might wish to consider $P(g)$ as shorthand notation for $\mathrm{Exp}_p(g)$ and always quantify just over Probs rather than Prevs in the constraints.[13]

[11] See Schervish et al. [49] for consideration of relaxation of this. We conjecture that the results of this paper still hold when we permit unbounded gambles, working instead in $\mathbb{R}^\Omega$ with the product topology. Moreover, since continuous linear functionals in $\mathbb{R}^\Omega$ are finitely supported, this suggests that one can restrict to just linear previsions, or probability functions, with finite support.

[12] A suggested interpretation following Levi [38, 39, 40] is to take credal judgements to be restrictions on the estimated value functions, or price functions, which the agent deems permissible to use in deliberation and evaluation. See Konek [31] for more on the inclusion of 'evaluation'.

[13] This is legitimate because of the isomorphism between linear previsions (of bounded gambles) and finitely additive probability functions.

To directly observe that the results of this paper work when one instead considers filters of probabilities, one needs to check firstly: that they generate linear previsions, so that the axioms on desirability hold; and secondly: that they have enough flexibility to play the role required in the separation results of Sublemmas 3.10.3 and 4.8.3.

For the remainder of the paper, then, we will understand our models of belief as collections of constraints on linear previsions, that is, $\mathcal{F} \subseteq \wp(\text{Prevs})$, and we say it is coherent when it a proper filter, as in Definition 2.2. Given the isomorphism between the frameworks there is no formal difference and all the results of Section 2 still hold, most of which anyway only depend on the filter structure.

We will keep using the terminology of probability filter and probability constraints as we wish to highlight the connection to the credal set model based on probability functions and the probabilistic underpinnings of the model.

3.1.1 Filter of lower previsions?

Every linear prevision has the property that $P(g) = -P(-g)$. Hence

$$[\![\text{Pr}(g) = -\text{Pr}(-g)]\!] = \{\text{Pr} \in \text{Prevs} \mid P(g) = -P(-g)\} = \text{Prevs}$$

and therefore this belongs to $\mathcal{F}$ for any coherent $\mathcal{F}$. That is, the agent should always hold the judgement which might be expressed as "my expected gain from buying the bet equals my expected loss from selling it", despite the absence of any fixed expected value or fair price for g.[14]

One could alternatively consider a filter not of linear previsions (equivalent to probabilities) but of lower previsions [see especially 59]. Lower previsions are themselves commonly used as a model of imprecise or indeterminate probability. Adopting a filter of lower previsions therefore has two distinct potential sources of imprecision: one arising from gaps in the agent's judgements, and another arising from allowing evaluation points to themselves be imprecise.

This would bring the framework closer to the approach of De Cooman et al. [16]. This is a question we will return to in Section 4.4. We here note, however, that using a filter of lower previsions will not make a difference to our study in Section 3. Filters of lower previsions generate the exact same collections of sets of desirable gambles as filters of probability functions do.[15]

It will, however, make a difference to determining choice functions, discussed in Section 4. I conjecture that this would result in dropping a component of the mixing axiom, axiom ($\text{K}_{\text{Mix-conv}}$), although that axiom ($\text{K}_{\text{Mix-cl}}$) would be retained. To drop that too, De Cooman et al. [16] consider a filter of coherent sets of desirable gambles directly. We will discuss this in Section 4.4.

[14] This should not be interpreted as saying that her upper and lower previsions are identical. In line with our proposed link with desirability of Section 3.2, we would most naturally understand the agent's lower prevision as the supremum price at which she judges buying the gamble desirable, and the upper prevision as the infimum price at which she judges selling the gamble desirable. In general, it may be that

$$\sup\{r \mid [\![\text{Pr}(g) > r]\!] \in \mathcal{F}\} < \inf\{r \mid [\![\text{Pr}(g) < r]\!] \in \mathcal{F}\}.$$

[15] To see this, we already show that any set of gambles satisfying the relevant axioms can be derived from a filter of linear previsions (Theorem 3.10, or in the more general settings, Theorems 3.14 and 3.17), so filters of lower previsions are at least as general. But one can also check that they satisfy the relevant axioms too. Note that axiom (D_{ArchCl}) just depends on continuity of the function.

3.2 Desirability judgements

The desirable gambles model represents uncertainty with a set of desirable gambles; those judged as preferable to the status quo. The standard axioms on coherence for a set of desirable gambles are as follows [68, 45]:

Definition 3.3. A set of gambles, $\mathcal{D} \subseteq \mathcal{G}$, is **coherent** if it satisfies the following axioms:

($D_{0\notin}$) $0 \notin \mathcal{D}$.

($D_{\gneqq 0}$) If $g \in \mathcal{G}_{\gneqq 0}$, then $g \in \mathcal{D}$.

(D_λ) If $g \in \mathcal{D}$ and $\lambda > 0$, then $\lambda g \in \mathcal{D}$.

(D_+) If $f, g \in \mathcal{D}$, then $f + g \in \mathcal{D}$.

We propose connecting the framework by understanding a desirable gamble as one where you judge it to have positive expected value. That is, we ask whether $[\![\mathrm{Pr}(g) > 0]\!] \in \mathcal{F}$.

Definition 3.4.

$$\mathcal{D}_\mathcal{F} = \{g \in \mathcal{G} \mid [\![\mathrm{Pr}(g) > 0]\!] \in \mathcal{F}\}.$$

We first note an equivalent formulation:

Proposition 3.5. *$g \in \mathcal{D}_\mathcal{F}$ iff there is some $C \in \mathcal{F}$ with $P(g) > 0$ for all $P \in C$*

Proof. If C is such that $P(g) > 0$ for all $p \in C$, then $C \subseteq [\![\mathrm{Pr}(g) > 0]\!]$. So by axiom ($F_\supseteq$), $C \in \mathcal{F}$ implies $[\![\mathrm{Pr}(g) > 0]\!] \in \mathcal{F}$, which implies that $g \in \mathcal{D}_\mathcal{F}$ by Definition 3.4.

If $[\![\mathrm{Pr}(g) > 0]\!] \in \mathcal{F}$, then we can put C as $[\![\mathrm{Pr}(g) > 0]\!]$, observing that $P(g) > 0$ for all $P \in C$. □

It is interesting to then note a special case of this result stated for principal filters (those of the form in Proposition 2.4):

Corollary 3.6. *If $\mathcal{F}$ has the form $\mathcal{F}_{C^*}$ for some $C^* \subseteq \mathrm{Prevs}$, that is, $\mathcal{F} = \{D \subseteq \mathrm{Prevs} \mid D \supseteq C^*\}$, then $g \in \mathcal{D}_\mathcal{F}$ iff $P(g) > 0$ for all $P \in C^*$.*

When the filter is principal it is equivalent to a credal set. If a credal set is understood as a set of linear previsions, then $\mathcal{D}_\mathcal{F}$ is just those gambles where $P(g) > 0$ for all linear previsions, P, in the credal set; if it is understood as a set of probability functions, it is when $\mathrm{Exp}_p(g) > 0$ for all probabilities, p, in the credal set. That is, it matches the usual definition [45, §1.6.2].

Having seen how we are relating the framework of probability filters to that of sets of desirable gambles framework, we will now turn to comparing the two frameworks. We will ask the following questions:

(i) Is every $\mathcal{D}_\mathcal{F}$ coherent?

(ii) Does the probability filter framework encompass that of coherent sets of desirable gambles? I.e., can every coherent $\mathcal{D}$ be obtained as $\mathcal{D}_\mathcal{F}$ for some $\mathcal{F}$?

(iii) Does the probability filter framework go beyond that of coherent sets of desirable gambles? I.e., are there distinct $\mathcal{F}$ and $\mathcal{F}'$ where $\mathcal{D}_{\mathcal{F}} = \mathcal{D}_{\mathcal{F}'}$?

We will answer 'yes' to all three of these questions (Theorems 3.7, 3.10 and 3.18). This will require just that $\mathcal{F}$ is coherent and pointwise regular.

As probability filters are just encoding sets of probability constraints which are closed (and consistent) under finitary probabilistic entailment, positive answers to our first two questions entail that the usual notion of coherence imposed on sets of desirable gambles exactly matches that of finitary probabilistic entailment, along with the additional assumption given by pointwise regularity.

3.3 Question (i). Every $\mathcal{D}_{\mathcal{F}}$ is coherent.

Requiring that $\mathcal{F}$ is coherent and pointwise regular ensures that the resultant desirability judgements are coherent; that is, for a coherent and pointwise regular $\mathcal{F}$, the corresponding set of desirable gambles, $\mathcal{D}_{\mathcal{F}}$, will satisfy the axioms for coherence of a set of desirable gambles.

Theorem 3.7. *If $\mathcal{F}$ is coherent and pointwise regular, then $\mathcal{D}_{\mathcal{F}}$ is a coherent set of desirable gambles.*

Proof. Axiom ($D_{0\notin}$) follows from axiom (F_{Proper}) by observing that $[\![\Pr(0) > 0]\!] = \varnothing$.

Axiom (D_λ) holds because when $\lambda > 0$, $[\![\Pr(g) > 0]\!] = [\![\Pr(\lambda g) > 0]\!]$.

Axiom (D_+): If $g \in \mathcal{D}_{\mathcal{F}}$ and $f \in \mathcal{D}_{\mathcal{F}}$, then $[\![\Pr(g) > 0]\!] \in \mathcal{F}$ and $[\![\Pr(f) > 0]\!] \in \mathcal{F}$. So $[\![\Pr(g) > 0 \text{ and } \Pr(f) > 0]\!] \in \mathcal{F}$ by axiom ($F_\cap$). If $P(g) > 0$ and $P(f) > 0$ then also $P(g + f) > 0$ by linearity. So $g + f \in \mathcal{D}_{\mathcal{F}}$, using Proposition 3.5.

Axiom ($D_{\gneqq 0}$): Suppose $g \in \mathcal{G}_{\gneqq 0}$. Consider some ω^* such that $g(\omega^*) > 0$. If P is a linear prevision with $P(\omega^*) > 0$, then $P(g) > 0$. By axiom ($F_{\text{reg-pw}}$), $[\![\text{pr}(\omega^*) > 0]\!] \in \mathcal{F}$, and therefore $[\![\Pr(g) > 0]\!] \supseteq [\![\text{pr}(\omega^*) > 0]\!]$ is also in $\mathcal{F}$ by axiom ($F_\supseteq$). ∎

3.4 Question (ii). Every coherent $\mathcal{D}$ can be obtained as $\mathcal{D}_{\mathcal{F}}$ for some $\mathcal{F}$.

We can also reverse the process: starting with a set of desirable gambles, $\mathcal{D}$, we can determine a probability filter $\mathcal{F}_{\mathcal{D}}$ which is the least committal filter evaluating each $g \in \mathcal{D}$ as desirable, i.e., with $[\![\Pr(g) > 0]\!] \in \mathcal{F}$ for each $g \in \mathcal{D}$.

Definition 3.8. For a set of gambles $\mathcal{D} \subseteq \mathcal{G}$, define $\mathcal{F}_{\mathcal{D}}$ by:

$$\mathcal{F}_{\mathcal{D}} := \text{ext}(\{[\![\Pr(g) > 0]\!] \mid g \in \mathcal{D}\}).$$

Recalling the definition of ext (Definition 2.10), we see that $C \in \mathcal{F}_{\mathcal{D}}$ iff there are finitely many gambles in $\mathcal{D}$, some $g_1, \ldots, g_n \in \mathcal{D}$, with

$$C \supseteq [\![\Pr(g_1) > 0]\!] \cap \ldots \cap [\![\Pr(g_n) > 0]\!].$$

That is,

$$C \supseteq \{P \in \text{Prevs} \mid P(g_1) > 0 \text{ and } \ldots \text{ and } P(g_n) > 0\}.$$

Before moving to our main result and its proof, we need some additional setup.

Setup 3.9. We endow $\mathcal{G}$ with the sup-norm or uniform norm topology, i.e., we are working with $\ell^\infty(\Omega)$. More precisely, $f \in \mathsf{closure}(A)$ iff there is a sequence $\langle g_n \rangle$ with each $g_n \in A$ where $\sup_{\omega\in\Omega} |f(\omega) - g_n(\omega)| \longrightarrow 0$ as $n \longrightarrow \infty$.

For two sets of gambles, A, B, define $A + B := \{a + b \mid a \in A, b \in B\}$ (Minkowski addition) and $A - B := \{a - b \mid a \in A, b \in B\}$.

The convex hull of a set of gambles, B, is denoted

$$\mathsf{conv}(B) := \{\textstyle\sum_{i=1}^n \lambda_i g_i \mid n \in \mathbb{N} \setminus \{0\}, \lambda_i > 0, \sum_i \lambda_i = 1, g_i \in B\}.$$

The positive linear hull of a set of gambles, B, is denoted

$$\mathsf{posi}(B) := \{\textstyle\sum_{i=1}^n \lambda_i g_i \mid n \in \mathbb{N} \setminus \{0\}, \lambda_i > 0, g_i \in B\}.$$

The cone of B, is denoted

$$\mathsf{cone}(B) := \{\textstyle\sum_{i=1}^n \lambda_i g_i \mid n \in \mathbb{N} \setminus \{0\}, \lambda_i \geq 0, g_i \in B\}.$$

For $B \neq \varnothing$, this is $\mathsf{posi}(B) \cup \{0\}$.

Within formal results, we will make use of

$$S_b := \mathsf{cone}(\{b\}) - \mathcal{G}_{\geq 0}.$$

Illustrations of some of these notions are given in Fig. 1.

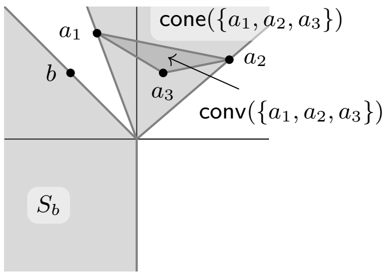

Figure 1: An example of S_b along with $\mathsf{conv}(\{a_1, a_2, a_3\})$ and $\mathsf{cone}(\{a_1, a_2, a_3\})$.

Our main result is the following:

Theorem 3.10. *If $\mathcal{D}$ is coherent, then $\mathcal{F}_\mathcal{D}$ is coherent and pointwise regular and $\mathcal{D} = \mathcal{D}_{\mathcal{F}_\mathcal{D}}$, i.e.:*

$$f \in \mathcal{D} \text{ iff } [\![\Pr(f) > 0]\!] \in \mathcal{F}_\mathcal{D}. \tag{2}$$

We will prove some sublemmas which form the main moving parts of the proof. Our first two sublemmas show why S_b is useful for our argument.

Sublemma 3.10.1.

$$S_b = (\mathsf{posi}(\{b\}) - \mathcal{G}_{\geq 0}) \ \cup \ \mathcal{G}_{\leq 0}.$$

Moreover, if $f \in \mathsf{posi}(\{b\}) - \mathcal{G}_{\geq 0}$, then $b \in \mathsf{posi}(\{f\}) + \mathcal{G}_{\geq 0}$.

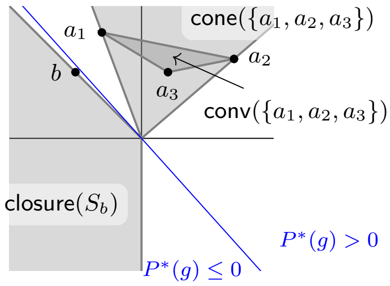

Figure 2: Illustrates the proof of Sublemma 3.10.3. As Ω is finite in this illustration, $\mathsf{closure}(S_b) = S_b$, which was drawn in Fig. 1. When $\mathsf{conv}(\{a_1, \dots, a_n\})$ and $\mathsf{closure}(S_b)$ are disjoint, a separating hyperplane can be found, which is > 0 on $\{a_1, \dots, a_n\}$ and ≤ 0 on $\mathsf{closure}(S_b)$.

Proof.

$$\begin{aligned} S_b &= \mathsf{cone}(\{b\}) \;-\; \mathcal{G}_{\geq 0} \\ &= (\mathsf{posi}(\{b\}) \cup \{0\}) \;-\; \mathcal{G}_{\geq 0} \\ &= (\mathsf{posi}(\{b\}) - \mathcal{G}_{\geq 0}) \;\cup\; (\{0\} - \mathcal{G}_{\geq 0}) \\ &= (\mathsf{posi}(\{b\}) - \mathcal{G}_{\geq 0}) \;\cup\; \mathcal{G}_{\leq 0}. \end{aligned}$$

If $f \in \mathsf{posi}(\{b\}) - \mathcal{G}_{\geq 0}$ then $f = \mu b - h$ with $\mu > 0$. Thus $b = {}^1\!/\!_\mu(f + h) = {}^1\!/\!_\mu f + {}^1\!/\!_\mu h$. As $\mu > 0$ also ${}^1\!/\!_\mu > 0$, so ${}^1\!/\!_\mu f \in \mathsf{posi}(\{f\})$ and ${}^1\!/\!_\mu h \in \mathcal{G}_{\geq 0}$, showing that $b \in \mathsf{posi}(\{f\}) + \mathcal{G}_{\geq 0}$. □

From which we can show:

Sublemma 3.10.2. *If $\mathcal{D}$ is coherent and $\mathcal{D} \cap S_b \neq \varnothing$, then $b \in \mathcal{D}$.*

Proof. We first note that for coherent $\mathcal{D}$, $f \in \mathcal{G}_{\leq 0}$ implies $f \notin \mathcal{D}$. This is a standard property of coherent sets of desirable gambles. We can prove it directly as follows: By axiom ($\mathrm{D}_{0\notin}$), $0 \notin \mathcal{D}$. For $f \in \mathcal{G}_{\leq 0} \setminus \{0\}$, $-f \in \mathcal{G}_{\gneq 0}$ so $-f \in \mathcal{D}$ by axiom ($\mathrm{D}_{\gneq 0}$). If also $f \in \mathcal{D}$ then $f + -f = 0 \in \mathcal{D}$ by axiom (D_{+}), contradicting axiom ($\mathrm{D}_{0\notin}$). Therefore, for all $f \in \mathcal{G}_{\leq 0}$, we have $f \notin \mathcal{D}$.

We can then use Sublemma 3.10.1 to see that for any $f \in \mathcal{D} \cap S_b$, it must be that $f \in \mathsf{posi}(\{b\}) - \mathcal{G}_{\geq 0}$ and therefore $b \in \mathsf{posi}(\{f\}) + \mathcal{G}_{\geq 0}$. So $b = \lambda f + h$ for some $h \in \mathcal{G}_{\geq 0}$ and $\lambda > 0$. By axiom (D_{λ}), $\lambda f \in \mathcal{D}$. If $h = 0$ then already we have $b \in \mathcal{D}$. Otherwise $h \in \mathcal{G}_{\gneq 0}$ in which case $h \in \mathcal{D}$ by axiom ($\mathrm{D}_{\gneq 0}$), and thus $b = \lambda f + h \in \mathcal{D}$ by axiom (D_{+}). □

The significant work done to prove our main theorem is in the next lemma, illustrated in Fig. 2.

Sublemma 3.10.3. *If $\mathsf{conv}(\{a_1, \dots, a_n\})$ and $\mathsf{closure}(S_b)$ are disjoint, then there is a linear prevision, P^* such that $P^*(a_i) > 0$ for all $a_1, \dots, a_n$ and $P^*(b) \leq 0$.*

This is proved by means of a separating hyperplane theorem.

Proof. We will make use of the version of the separating hyperplane theorem given in Theorem 2.5 of Klee [30].[16]

Recall that we are working with the space of gambles with the sup-norm topology, i.e., $\ell^\infty(\Omega)$, which is a locally convex topological linear space.

Assume that $\mathsf{conv}(\{a_1, \ldots, a_n\})$ and $\mathsf{closure}(S_b)$ are disjoint, as in the assumption of the theorem.

$S_b = \mathsf{cone}(\{b\}) - \mathcal{G}_{\geq 0}$, with both $\mathsf{cone}(\{b\})$ and $\mathcal{G}_{\geq 0}$ being convex cones, so it is also a convex cone. Thus, $\mathsf{closure}(S_b)$ is a closed convex cone.

$\mathsf{conv}(\{a_1, \ldots, a_n\})$ can be extended to $\mathsf{cone}(\{a_1, \ldots, a_n\})$, which is a locally compact closed convex cone. By assumption that $\mathsf{conv}(\{a_1, \ldots, a_n\})$ is disjoint from $\mathsf{closure}(S_b)$, $\mathsf{cone}(\{a_1, \ldots, a_n\})$ and $\mathsf{closure}(S_b)$ only intersect at 0.

So by the separating hyperplane theorem for cones of Klee [30, Theorem 2.5], we can find a continuous linear functional T such that $T(g) > 0$ for all g where $g \in \mathsf{posi}(\{a_1, \ldots, a_n\})$ and $-g \notin \mathsf{posi}(\{a_1, \ldots, a_n\})$, and $T(f) \leq 0$ for all $f \in \mathsf{closure}(S_b)$.

Since $0 \in S_b$, by our disjointness assumption, $0 \notin \mathsf{conv}(\{a_1, \ldots, a_n\})$ and therefore also $0 \notin \mathsf{posi}(\{a_1, \ldots, a_n\})$. This ensures that for all g with $g \in \mathsf{posi}(\{a_1, \ldots, a_n\})$, we have $-g \notin \mathsf{posi}(\{a_1, \ldots, a_n\})$, otherwise we would have that $0 \in \mathsf{posi}(\{a_1, \ldots, a_n\})$. Therefore, in fact $T(g) > 0$ for all $g \in \mathsf{posi}(\{a_1, \ldots, a_n\})$, and thus, in particular, for $a_1, \ldots, a_n$.

For any $g \geq 0$, $-g \in S_b$, and thus $T(-g) \leq 0$ so $T(g) \geq 0$. Thus T is a positive linear functional and therefore a monotone linear functional.

It remains to show that T can be normalised for which we require that $T(\mathbf{1}_\Omega) > 0$. Note that we already have that $T(a_1) > 0$. As a_1 is bounded and non-zero, $\sup_{\omega \in \Omega} |a_1(\omega)|$ is finite and non-zero. Put $\lambda := 1/\sup_{\omega\in\Omega}|a_1(\omega)| > 0$. Then observe that $\lambda a_1(\omega) \leq 1$ for each $\omega \in \Omega$, so $\lambda a_1 \leq \mathbf{1}_\Omega$. Since T is a monotone linear functional, then $T(\mathbf{1}_\Omega) \geq T(\lambda a_1) = \lambda T(a_1) > 0$. Thus, T can be normalised, and once T is normalised, it is our required linear prevision. □

These sublemmas can be combined fruitfully in the case where S_b is closed. This is guaranteed when Ω is finite, as will be shown in Sublemma A.1.2, but since we are interested in the case where Ω may be infinite we need to impose an alternative restriction to ensure it is closed (Example 3.16). We can show that when $b \notin \mathcal{G}_{\gneq 0}$ then it is closed.

Sublemma 3.10.4. *If $b \notin \mathcal{G}_{\gneq 0}$, then S_b is closed.*

Proof. Observe that $\mathcal{G}_{\geq 0}$ is a closed convex cone, and $\mathsf{cone}(\{b\})$ is a convex cone which is finitely generated and thus closed and locally compact.

If $b \notin \mathcal{G}_{\gneq 0}$, then $\mathcal{G}_{\geq 0}$ and $\mathsf{cone}(\{b\})$ only intersect at 0, and thus, by Theorem 2.1 of Klee [30], $S_b = \mathsf{cone}(\{b\}) - \mathcal{G}_{\geq 0}$ is closed. □

We now show how these sublemmas can be combined to construct a proof of Theorem 3.10.

[16] One can alternatively prove this with a more standard separation theorem than the one for cones. For example, using Schechter [48, HB19] we can separate $\mathsf{conv}(\{a_1, \ldots, a_n\})$ from $\mathsf{closure}(S_B)$, obtaining a linear functional T such that $\min\{T(a_i) \mid i = 1, \ldots, n\} > \sup\{T(f) \mid f \in \mathsf{closure}(S_B)\}$. To show that T is monotone, one then additionally needs to show that $\sup\{T(f) \mid f \in \mathsf{closure}(S_B)\} \leq 0$. If $T(f) > 0$ then put $\lambda > \frac{T(a_1)}{T(f)} > 0$ and note that $T(\lambda f) = \lambda T(f) > T(a_1)$. Moreover, $\lambda f \in \mathsf{closure}(S_B)$, as it is a cone; contradicting the separation. Thanks to Arthur Van Camp for the reference and discussion.

Proof of Theorem 3.10. Assume $\mathcal{D}$ is coherent. We will first show that $\mathcal{D} = \mathcal{D}_{\mathcal{F}_\mathcal{D}}$, i.e, Eq. (2), before checking that $\mathcal{F}_\mathcal{D}$ is coherent and pointwise regular.

For any $b \in \mathcal{D}$ we have $[\![\Pr(b) > 0]\!] \in \mathcal{F}_\mathcal{D}$ by construction. The interesting direction is the converse, that is, if $[\![\Pr(b) > 0]\!] \in \mathcal{F}_\mathcal{D}$ then $b \in \mathcal{D}$.

By definition of $\mathcal{F}_\mathcal{D}$, if $[\![\Pr(b) > 0]\!] \in \mathcal{F}_\mathcal{D}$ then there are some $a_1, \ldots, a_n \in \mathcal{D}$ where $[\![\Pr(b) > 0]\!] \supseteq [\![\Pr(a_1) > 0]\!] \cap \ldots \cap [\![\Pr(a_n) > 0]\!]$. We need to show that this entails that $b \in \mathcal{D}$.

By Sublemma 3.10.3, $\mathsf{conv}(\{a_1, \ldots, a_n\})$ and $\mathsf{closure}(S_b)$ must be non-disjoint, otherwise we would find some linear prevision violating the superset claim.

If $b \in \mathcal{G}_{\gneq 0}$, then $b \in \mathcal{D}$ by axiom ($\mathrm{D}_{\gneq 0}$).

Otherwise $b \notin \mathcal{G}_{\gneq 0}$. In which case, by Sublemma 3.10.4, S_b is closed, and thus in fact $\mathsf{conv}(\{a_1, \ldots, a_n\})$ and S_b are non-disjoint. Let $f \in \mathsf{conv}(\{a_1, \ldots, a_n\}) \cap S_b$. Then $f \in \mathcal{D}$ by axioms (D_λ) and (D_+), showing that $\mathcal{D} \cap S_b \neq \varnothing$. By Sublemma 3.10.2, then $b \in \mathcal{D}$, as required.

We have thus shown that if $\mathcal{D}$ is coherent and there are some $a_1, \ldots, a_n \in \mathcal{D}$ where $[\![\Pr(b) > 0]\!] \supseteq [\![\Pr(a_1) > 0]\!] \cap \ldots \cap [\![\Pr(a_n) > 0]\!]$ then $b \in \mathcal{D}$, as required. This gives us that $\mathcal{D} = \mathcal{D}_{\mathcal{F}_\mathcal{D}}$, i.e., Eq. (2).

It remains to show that $\mathcal{F}_\mathcal{D}$ is coherent and pointwise regular.

By definition of ext, $\mathcal{F}_\mathcal{D}$ satisfies axioms ($\mathrm{F}_\cap$) and ($\mathrm{F}_\supseteq$). By axiom ($\mathrm{D}_{\gneq 0}$), $\mathcal{D} \neq \varnothing$, so $\mathcal{F}_\mathcal{D}$ satisfies axiom ($\mathrm{F}_{\neq\varnothing}$). To observe that $\mathcal{F}_\mathcal{D}$ satisfies axiom ($\mathrm{F}_{\mathrm{Proper}}$), observe that $\varnothing = [\![\Pr(0) > 0]\!] \notin \mathcal{F}_\mathcal{D}$, by our proved Eq. (2) and by axiom ($\mathrm{D}_{0\notin}$) giving us $0 \notin \mathcal{D}$.

For pointwise regularity, since $\mathbf{1}_\omega \in \mathcal{G}_{\gneq 0}$ by axiom ($\mathrm{D}_{\gneq 0}$), by Eq. (2), $[\![\Pr(\mathbf{1}_\omega) > 0]\!] = [\![\mathrm{pr}(\omega) > 0]\!] \in \mathcal{F}_\mathcal{D}$. □

An immediate corollary of Theorem 3.10 is:

Corollary 3.11. *For distinct coherent $\mathcal{D}$ and $\mathcal{D}'$, $\mathcal{F}_\mathcal{D}$ and $\mathcal{F}_{\mathcal{D}'}$ are distinct.*

Proof. This follows immediately from Eq. (2). □

This tells us that the probability filter framework is at least as informative as that of coherent set of desirable gambles model. In fact it will go beyond this, which we will discuss in the next section. However first it is illustrative to consider an example of a coherent set of gambles which is not representable by a single set of probabilities. By Theorem 3.10, however, it is representable by a probability filter.

Example 3.12. Fix $\Omega = \{\mathrm{H}, \mathrm{T}\}$, representing the outcomes of a coin toss. Consider $\mathcal{D}^*$ given by $g \in \mathcal{D}^*$ iff $g(\mathrm{H}) + g(\mathrm{T}) > 0$, or $g(\mathrm{H}) = -g(\mathrm{T})$ and $g(\mathrm{H}) > 0$. This is illustrated in Fig. 3.

In particular, we have:

$$b := \langle 0.5, -0.5 \rangle \in \mathcal{D}^*.$$

$$a_\epsilon := \langle -0.5, 0.5 \rangle + \langle \epsilon, \epsilon \rangle = \langle -0.5 + \epsilon, 0.5 + \epsilon \rangle \in \mathcal{D}^* \text{ for } \epsilon \text{ any positive real.}$$

So, for example $a_{0.001} = \langle -0.499, 0.501 \rangle \in \mathcal{D}^*$.

Observe that $P(b) > 0$ iff $P(\mathrm{H}) > 0.5$. And that $P(a_\epsilon) > 0$ iff $P(\mathrm{H}) < 0.5 + \epsilon$. We thus see that there is no $P^* \in \mathrm{Prevs}$ where $P^*(g) > 0$ for all $g \in \mathcal{D}^*$. (See also [45, p.20].)

$\mathcal{D}^*$ is, however, a coherent set of desirable gambles, and it can be represented in the probability filter framework due to Theorem 3.10. $\mathcal{F}_{\mathcal{D}^*}$ will have that

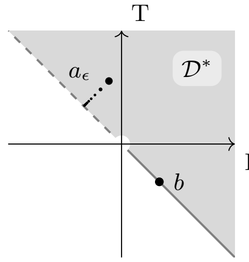

Figure 3: A coherent set of desirable gambles given by $g \in \mathcal{D}^*$ iff $g(\mathrm{H})+g(\mathrm{T}) > 0$ or $g(\mathrm{H}) = -g(\mathrm{T})$ and $g(\mathrm{H}) > 0$.

$[\![\Pr(b) > 0]\!] = [\![\Pr(\mathrm{H}) > 0.5]\!] \in \mathcal{F}_{\mathcal{D}^*}$ and $[\![\Pr(a_\epsilon) > 0]\!] = [\![\Pr(\mathrm{H}) < 0.5+\epsilon]\!] \in \mathcal{F}_{\mathcal{D}^*}$ for each positive real ϵ. It can in fact then be checked that $\mathcal{F}_{\mathcal{D}^*}$ is identical to $\mathcal{F}_{\mathrm{InfBiased}}$, as in Example 2.6.

3.5 Without regularity

Before moving to discuss question (iii), we discuss what can be maintained of our results if the pointwise regularity of the probability filters is dropped.

In this case, axiom ($\mathrm{D}_{\gneq 0}$) must also be dropped, at least if we keep the identification of desirability from filters in accordance with Definition 3.4, that is, as the judgement that a gamble has positive expected payout. We are able to obtain amended versions of our results Theorems 3.7 and 3.10. What this then provides is a characterisation of the axioms on desirable gambles which matches consistency and closure under finitary probabilistic entailment amongst judgements of gambles having positive expected value. We leave the proofs of all the results in this section to the appendix.

In the case of finite Ω, the axioms we will need on desirable gambles are the remaining axioms, ($\mathrm{D}_{0\notin}$), (D_λ), (D_+), and two additional axioms to replace axiom ($\mathrm{D}_{\gneq 0}$):

($\mathrm{D}_{\inf>0}$) If $\inf(g) > 0$, then $g \in \mathcal{D}$.

($\mathrm{D}_\geq$) If $g \in \mathcal{D}$ and $f \geq g$, then $f \in \mathcal{D}$.

See Van Camp and Seidenfeld [62, §2.3] for a discussion of axiom ($\mathrm{D}_\geq$).

Theorem 3.13. *If $\mathcal{F}$ is coherent, then $\mathcal{D}_\mathcal{F}$ satisfies axioms* ($\mathrm{D}_{0\notin}$), (D_λ), (D_+) ($\mathrm{D}_{\inf>0}$) *and* ($\mathrm{D}_\geq$)

When Ω is finite, we can also show that these axioms are sufficient for representation by a probability filter.

Theorem 3.14. *Suppose Ω is finite. If $\mathcal{D}$ satisfies axioms* ($\mathrm{D}_{0\notin}$), (D_λ), (D_+) ($\mathrm{D}_{\inf>0}$) *and* ($\mathrm{D}_\geq$), *then $\mathcal{F}_\mathcal{D}$ is coherent and $\mathcal{D} = \mathcal{D}_{\mathcal{F}_\mathcal{D}}$.*

This works when Ω is finite because then S_b is guaranteed to be closed.

When Ω is infinite, however, this will not work and we will need to add an extra axiom on desirability to get the frameworks to match.

To get representation of $\mathcal{D}$ by a probability filter, the Archimedean axiom would suffice [13, Def.22].

(D_{Arch}) If $f \in \mathcal{D}$ then there is some $\epsilon > 0$ such that $f - \epsilon \in \mathcal{D}$.

We are here using ϵ as the constant gamble returning a value of ϵ in every ω.

However, it is too strong. There are coherent $\mathcal{F}$ where $\mathcal{D}_{\mathcal{F}}$ does not satisfy axiom (D_{Arch}). Axiom (D_{Arch}) implies that $\mathcal{D}$ is an open set, which is not true for all coherent $\mathcal{D}$ (Example 3.12), which, as we have already seen, are representable by probability filters due to Theorem 3.10. Instead, we propose imposing a weaker axiom:

(D_{ArchCl}) If $f \in \mathcal{D}$ and for all $\epsilon > 0$, $g \in \mathsf{posi}(\{f - \epsilon\}) + \mathcal{G}_{\geq 0}$, then $g \in \mathcal{D}$.

Unlike axiom (D_{Arch}), this is satisfied on all $\mathcal{D}_{\mathcal{F}}$:

Theorem 3.15. *If $\mathcal{F}$ is coherent, then $\mathcal{D}_{\mathcal{F}}$ satisfies axiom* (D_{ArchCl})

Theorem 3.15 along with our forthcoming Theorem 3.17 show us that these axioms are exactly what's needed to characterise judgements of gambles having positive expected value arising from probability filters, i.e., being closed and consistent under finite probabilistic entailment. When Ω is finite, it can be simplified by dropping axiom (D_{ArchCl}), as shown in Theorems 3.13 and 3.14.

When Ω is finite, or when axiom ($D_{\gneq 0}$) is imposed, axiom (D_{ArchCl}) follows from the remaining axioms (as a consequence of Theorems 3.10, 3.14 and 3.15). But not when we drop axiom ($D_{\gneq 0}$) and allow Ω to be infinite.

Example 3.16. Let $\Omega = \mathbb{N}$ and consider gamble a given by $a(n) = 1/n$ for all $n > 0$ and $a(0) = 0$. Let $\mathcal{D} = \mathsf{posi}(\{a\}) + \mathcal{G}_{\geq 0}$. This satisfies axioms ($D_{0\notin}$), (D_{λ}) and (D_{+}) and axioms ($D_{\inf>0}$) and ($D_{\geq}$) but not axiom (D_{ArchCl}).

To see this, consider gamble b given by $b(n) = 1/n^2$ for all $n > 0$ and $b(0) = 0$. In Section A.4 we will show that $b \notin \mathcal{D}$ but $b \in \mathsf{posi}(\{a - \epsilon\}) + \mathcal{G}_{\geq 0}$ for any $\epsilon > 0$.

This is also an example where S_b is not closed, as $a - \epsilon \in S_b$ for each ϵ but $a \notin S_b$.

Having shown that axiom (D_{ArchCl}) is not too strong to match the notion of desirability obtained by probability filters, we can then see that it is strong enough to get representation by a probability filter, showing that these are exactly the axioms on desirability judgements derived from coherent probability filters.

Theorem 3.17. *If $\mathcal{D}$ satisfies axioms* ($D_{0\notin}$)*,* (D_{λ})*,* (D_{+}) ($D_{\inf>0}$)*,* ($D_{\geq}$) *and* (D_{ArchCl})*, then $\mathcal{F}_{\mathcal{D}}$ is coherent and $\mathcal{D} = \mathcal{D}_{\mathcal{F}_{\mathcal{D}}}$.*

This works because it allow us to move directly from $\mathcal{D} \cap \mathsf{closure}(S_b) \neq \varnothing$ to that $b \in \mathcal{D}$.

3.6 Question (iii). Probability filters go beyond desirability of gambles.

The probability filter model goes beyond the model of desirable gambles, as shown by the next theorem.

Theorem 3.18. *There are distinct coherent probability filters $\mathcal{F}$ and $\mathcal{F}'$ where $\mathcal{D}_{\mathcal{F}} = \mathcal{D}_{\mathcal{F}'}$.*

Proof. Take any non-convex $C^* \subseteq \text{Prevs}$, e.g., $[\![\Pr(H) = 0.2 \text{ or } \Pr(H) = 0.3]\!]$, and let B be its convex closure, which would be $[\![0.2 \leq \Pr(H) \leq 0.3]\!]$.

Then $P(g) > 0$ for all $P \in C^*$ implies also $P(g) > 0$ for any $P \in B$.

And so, $\mathcal{D}_{\mathcal{F}_{C^*}} = \mathcal{D}_{\mathcal{F}_B}$ by Corollary 3.6.

However, $\mathcal{F}_{C^*} \neq \mathcal{F}_B$, for example, $B \notin \mathcal{F}_{C^*}$. □

This argument makes use of probability filters which are characterised by non-convex probability constraints. In Section 4.7 we will provide an example of two probability filters which are both characterised by convex probability constraints but which result in the same set of desirable gambles.

In line with Section 2.4, if one wished to restrict the expressive power of the probability filter framework to make it equivalent to that of desirable gambles, one could add an additional axiom which ensures that $\mathcal{F}$ is characterised by its judgements on probability constraints of the form $[\![\Pr(g) > 0]\!]$. This is given by the following axiom:

($\text{F}_\mathcal{D}$) If $C \in \mathcal{F}$ then there are some finitely many gambles $g_1, \ldots, g_n$ with $[\![\Pr(g_i) > 0]\!] \in \mathcal{F}$ for each g_i and

$$C \supseteq [\![\Pr(g_1) > 0]\!] \cap \ldots \cap [\![\Pr(g_n) > 0]\!].$$

This axiom ensures that $\mathcal{F}$ is determined by its treatment of the probability constraints of the form $[\![\Pr(g) > 0]\!]$. By Theorem 3.18 we know that this won't hold for all coherent $\mathcal{F}$, but we can guarantee this if we restrict to the probability filters that satisfy axiom ($\text{F}_\mathcal{D}$).

Proposition 3.19. *For every coherent $\mathcal{D}$, $\mathcal{F}_\mathcal{D}$ satisfies axiom* ($\text{F}_\mathcal{D}$)*. Also, if $\mathcal{F}$ is a probability filter satisfying axiom* ($\text{F}_\mathcal{D}$) *then $\mathcal{F} = \mathcal{F}_{\mathcal{D}_\mathcal{F}}$.*

Proof. To show that every $\mathcal{F}_\mathcal{D}$ satisfies axiom ($\text{F}_\mathcal{D}$), we see by the definition of $\mathcal{F}_\mathcal{D}$ (Definition 3.8) that $C \in \mathcal{F}_\mathcal{D}$ iff there are finitely many $g_1, \ldots, g_n \in \mathcal{D}$ with

$$C \supseteq [\![\Pr(g_1) > 0]\!] \cap \ldots \cap [\![\Pr(g_n) > 0]\!].$$

And observe that $[\![\Pr(g_i) > 0]\!] \in \mathcal{F}_\mathcal{D}$, so we have axiom ($\text{F}_\mathcal{D}$).

To show that $\mathcal{F} = \mathcal{F}_{\mathcal{D}_\mathcal{F}}$, we also consult the definition of $\mathcal{D}_\mathcal{F}$ (Definition 3.4) and see that

$$C \in \mathcal{F}_{\mathcal{D}_\mathcal{F}} \text{ iff } \quad \begin{array}{l} \text{there are some } g_1, \ldots, g_n \\ \text{with } [\![\Pr(g_i) > 0]\!] \in \mathcal{F} \text{ for each } i \\ \text{and } C \supseteq [\![\Pr(g_1) > 0]\!] \cap \ldots \cap [\![\Pr(g_n) > 0]\!]. \end{array}$$

So axiom ($\text{F}_\mathcal{D}$) exactly guarantees that $\mathcal{F} \subseteq \mathcal{F}_{\mathcal{D}_\mathcal{F}}$. We also have $\mathcal{F}_{\mathcal{D}_\mathcal{F}} \subseteq \mathcal{F}$ just by Proposition 2.7. □

3.7 Updating and Conditionalisation

We presented conditionalisation in terms of probabilities and the ratio formula. When considered as linear previsions, this corresponds to:

Definition 3.20. For $P(\mathbf{1}_E) > 0$, $P(g \mid E) := \frac{P(\mathbf{1}_E g)}{P(\mathbf{1}_E)}$.

This is equivalent to Bayesian conditionalisation as in Section 2.6 as $\mathbf{1}_E\mathbf{1}_F = \mathbf{1}_{E\cap F}$. Observe the following:

Proposition 3.21. *Suppose $P(\mathbf{1}_E) > 0$. Then $P(g \mid E) > 0$ iff $P(\mathbf{1}_E g) > 0$.*

Proof. Follows immediately from the definition. □

We defined conditionalisation in Section 2.6 which we can apply also to the linear previsions setting to determine $\mathcal{F}|E$ from $\mathcal{F}$. We repeat the definition now applied to previsions:

Definition 3.22.

$$\mathcal{F}|E = \left\{ D \subseteq \text{Prevs} \;\middle|\; \begin{array}{c} \text{there is } C \in \mathcal{F} \text{ and } C \subseteq [\![\Pr(\mathbf{1}_E) > 0]\!] \\ \text{with } D \supseteq \{P(\cdot \mid E) \mid P \in C \text{ and } P(\mathbf{1}_E) > 0\} \end{array} \right\}$$

A natural question is then how $\mathcal{D}_{\mathcal{F}|E}$ relates to $\mathcal{D}_{\mathcal{F}}$? We can show that it is characterised by one's prior judgements of the called-off gambles. That is, it matches exactly the notion of E-desirability of a gamble as given by Walley [67].

Proposition 3.23. *Suppose $\mathcal{F}$ is coherent and pointwise regular. Then $g \in \mathcal{D}_{\mathcal{F}|E}$ iff $\mathbf{1}_E g \in \mathcal{D}_{\mathcal{F}}$.*

Proof. We are required to show that $[\![\Pr(g) > 0]\!] \in \mathcal{F}|E$ iff $[\![\Pr(\mathbf{1}_E g) > 0]\!] \in \mathcal{F}$.

$$\begin{aligned} [\![\Pr(g) > 0]\!] \in \mathcal{F}|E \text{ iff } & \text{there is some } C \in \mathcal{F} \text{ such that} \\ & \quad P(g \mid E) > 0 \text{ for all } P \in C \text{ with } P(\mathbf{1}_E) > 0. \\ \text{iff } & \text{there is some } C \in \mathcal{F} \text{ such that} \\ & \quad P(\mathbf{1}_E g) > 0 \text{ for all } P \in C \text{ with } P(\mathbf{1}_E) > 0. \end{aligned}$$

If $[\![\Pr(\mathbf{1}_E g) > 0]\!] \in \mathcal{F}$ then this is exactly such a C, as $P(\mathbf{1}_E) > 0$ for all P with $P(\mathbf{1}_E g) > 0$. This gives us that $[\![\Pr(\mathbf{1}_E g) > 0]\!] \in \mathcal{F}$ implies $[\![\Pr(g) > 0]\!] \in \mathcal{F}|E$.

For the converse: suppose there is such a C, i.e., $C \in \mathcal{F}$ where $P(\mathbf{1}_E g) > 0$ for all $P \in C$ with $P(\mathbf{1}_E) > 0$. Then for every $P \in C \cap [\![\Pr(\mathbf{1}_E) > 0]\!]$, $P(\mathbf{1}_E) > 0$ and so $P(\mathbf{1}_E g) > 0$, so $C \cap [\![\Pr(\mathbf{1}_E) > 0]\!] \subseteq [\![\Pr(\mathbf{1}_E g) > 0]\!]$. By pointwise regularity, $[\![\Pr(\mathbf{1}_E) > 0]\!] \in \mathcal{F}$, and therefore $C \cap [\![\Pr(\mathbf{1}_E) > 0]\!] \in \mathcal{F}$ giving us that $[\![\Pr(\mathbf{1}_E g) > 0]\!] \in \mathcal{F}$ by axiom ($\mathrm{F}_{\supseteq}$). □

Note that $\mathcal{D}_{\mathcal{F}|E}$ violates axiom ($\mathrm{D}_{\gneq 0}$). This is not a conflict with Theorem 3.7 because the updated $\mathcal{F}|E$ is not pointwise regular. Instead, it is merely pointwise regular *on E*. The resultant set of gambles will be irregularly coherent in the sense of Theorem 3.13.

In work on conditioning for sets of desirable gambles [see 15, §3.2], authors restrict to just those gambles defined on E. To keep the conditionalisation on filters as a generalisation of Bayesian conditioning of probabilities or linear previsions, where it is standard to not change the space of possible outcomes, we have not pursued this.

4 Probability filters and choice functions

4.1 Choice functions

Choice functions provide a more general model of uncertainty than coherent sets of desirable gambles, which can be seen as restricting attention to binary choice [53]. They have been studied in detail by Gert de Cooman, Jasper De Bock and Arthur Van Camp [12, 13, 60, 16], whose formalism of the choice functions model is to consider which gamble *sets* are desirable instead of just which individual gambles are desirable. This clearly extends the model of desirable gambles just considered. We adopt their approach as our understanding of the model of choice functions. The connection to choice is spelled out in [12].

We linked a judgement of a gamble as desirable to the credal judgement that the expected payout of the gamble is strictly positive. What does a gamble set being judged as desirable correspond to? It is glossed as that some member of the set is desirable, but there are two ways that this can be considered and linked to the probability filter framework.

We propose understanding that a gamble set is desirable for an agent whose belief state is captured by $\mathcal{F}$ just if she is committed to the probability constraint that some member of the set has positive expected value. That is, when $[\![\text{there is some } g \in A \text{ with } \Pr(g) > 0]\!] \in \mathcal{F}$. There need not need to be any individual member of the set which she judges to have positive expected value. This is the association that we will adopt for this paper.

Definition 4.1. For a probability filter, $\mathcal{F}$, define a set of gamble sets $\mathcal{K}_{\mathcal{F}}$ by:

$$\mathcal{K}_{\mathcal{F}} := \{B \subseteq \mathcal{G} \mid [\![\text{there is some } g \in B \text{ with } \Pr(g) > 0]\!] \in \mathcal{F}\}$$

If one then uses these assessments to rationalise the behaviour in choice-theoretic scenarios in the way described in De Bock and De Cooman [12], this is to apply Levi's rule of E-admissibility [36, 37], or rather a generalisation of it which applies to all probability filters rather than just those that arise from credal sets.

There are alternative prominent decision procedures for credal sets, in particular Maximality [67, 55]. To instead obtain the recommendations in accordance with Maximality by still using the same association as in [12], we could instead adopt an alternative definition:

$$\mathcal{K}_{\mathcal{F}}^{\text{Max}} := \{B \subseteq \mathcal{G} \mid \text{there is some } g \in B \text{ with } [\![\text{pr}(g) > 0]\!] \in \mathcal{F}\}$$

This requires that some particular member of the set must be such that you judge it as having positive expected value, as opposed to holding the judgement that some member has positive expected value but not being able to commit to any particular one.

In this paper, we will continue to work with $\mathcal{K}_{\mathcal{F}}$ based on E-admissibility. This is in line with Seidenfeld et al. [53] who is focused on choice functions derived from credal sets by E-admissibility. De Cooman, De Bock and Van Camp [60, 12, 13, 16] instead wish to provide a notion of coherence for choice functions which will allow for both those choice functions derived from Maximality and those derived from E-admissibility as coherent. We will discuss this and their more general framework in Section 4.4.

We have now specified how we determine a set of desirable gamble sets from a given probability filter: we have adopted Definition 4.1, generalising E-admissibility to the probability filter framework.

We now present our key questions, paralleling those we asked in the desirable gambles framework.

(i) Is every $\mathcal{K}_{\mathcal{F}}$ coherent?

(ii) Does the probability filter framework encompass the choice functions framework? Can every coherent $\mathcal{K}$ be obtained as $\mathcal{K}_{\mathcal{F}}$ for some coherent probability filter, $\mathcal{F}$?

(iii) Does the probability filter framework go beyond that of coherent sets of desirable gamble sets? I.e., are there distinct $\mathcal{F}$ and $\mathcal{F}'$ where $\mathcal{K}_{\mathcal{F}} = \mathcal{K}_{\mathcal{F}'}$?

To give answers to these questions we will need to specify a notion of coherence for sets of desirable gamble sets, which is what we turn to in the next section.

4.2 Question (i). Every $\mathcal{K}_{\mathcal{F}}$ is coherent.

The axioms that we impose for coherence of sets of desirable gamble sets are essentially an instance of the general axioms for sets of desirable sets of things of de Cooman et al. [16, §2.3]. We will merely impose what these authors call *finite* coherence, but we simply call it coherence.

Note that in this, and as in de Cooman et al. [16, §2.3], we are considering also infinite sets of gambles to be evaluated, unlike Van Camp [60], De Bock and de Cooman [12, 13].

Definition 4.2. $\mathcal{K} \subseteq \wp(\mathcal{G})$ is **coherent** if it satisfies

($\mathrm{K}_{\varnothing}$) $\varnothing \notin \mathcal{K}$

($\mathrm{K}_{\supseteq}$) If $A \in \mathcal{K}$ and $B \supseteq A$, then $B \in \mathcal{K}$

($\mathrm{K}_{\setminus\{0\}}$) If $A \in \mathcal{K}$ then $A \setminus \{0\} \in \mathcal{K}$.

($\mathrm{K}_{\gneq 0}$) If $g \in \mathcal{G}_{\gneq 0}$, then $\{g\} \in \mathcal{K}$.

($\mathrm{K}_{\mathrm{conv}}$) If $A_1, \ldots, A_n \in \mathcal{K}$ and for each sequence $\langle g_1, \ldots, g_n\rangle \in A_1 \times \ldots \times A_n$, $f_{\langle g_1,\ldots,g_n\rangle}$ is some member of $\mathsf{conv}(\{g_1, \ldots, g_n\})$, then $\{f_{\langle g_1,\ldots,g_n\rangle} | \langle g_1, \ldots, g_n\rangle \in A_1 \times \ldots \times A_n\} \in \mathcal{K}$.

($\mathrm{K}_{\mathrm{scalar}}$) If $A \in \mathcal{K}$ and for each $g \in A$, $\lambda_g > 0$ then $\{\lambda_g g \mid g \in A\} \in \mathcal{K}$.

($\mathrm{K}_{\geq}$) If $A \in \mathcal{K}$ and for each $g \in A$, f_g is some gamble where $f_g \geq g$, then $\{f_g \mid g \in A\} \in \mathcal{K}$.

To stay closer to the formulation of de Cooman et al. [16, Axiom $\mathrm{K}_5^{\mathrm{fin}}$], we could instead combine axioms ($\mathrm{K}_{\mathrm{conv}}$) to ($\mathrm{K}_{\geq}$) into a single axiom that has the same form as axiom ($\mathrm{K}_{\mathrm{conv}}$) but replaces $\mathsf{posi}(\{g_1, \ldots, g_n\})$ with $\mathsf{posi}(\{g_1, \ldots, g_n\}) + \mathcal{G}_{\geq 0}$. Or even to also combine it with axiom ($\mathrm{K}_{\gneq 0}$) and use $\mathsf{posi}(\{g_1, \ldots, g_n\} \cup \mathcal{G}_{\gneq 0})$. We have elected to keep the axioms separate as we find it more perspicuous to be able to individually refer to them in our proofs. It is also advantageous to keep axiom ($\mathrm{K}_{\gneq 0}$) as a separate axiom as it will be all that is changed when we consider probability filters which need not be pointwise regular Section 4.6.

Using these axioms, one can give an affirmative answer to our first question, i.e., every $\mathcal{K}_\mathcal{F}$ is coherent. In fact, this can be seen as a consequence of the general results of de Cooman et al. [16, Theorem 25] (which were based on an early version of this paper, Campbell-Moore [6]), which show that coherent sets of desirable gamble sets can be obtained as filters of coherent sets of desirable gambles. By adopting Definition 4.1, the choice functions derived from probability filters are exactly those derived from filters of certain kinds of coherent sets of desirable gambles, those obtained from probability functions. Since they show that all filters of coherent sets of desirable gambles generate coherent choice functions, these do too.

We however include a direct proof because of its importance for this paper. In fact, we also include our further axiom, axiom ($\mathrm{K}_{\gneq 0}$) which De Cooman et al. [16] do not, at least in the instance of the results they consider, although it would nonetheless follow from their more general results.

Proposition 4.3. *If $\mathcal{F}$ is coherent and pointwise regular, then $\mathcal{K}_\mathcal{F}$ is coherent.*

Proof. The definition of $\mathcal{K}_\mathcal{F}$ is that $B \in \mathcal{K}_\mathcal{F}$ iff $[\![\exists h \in B : \Pr(h) > 0]\!] \in \mathcal{F}$. We check each of the axioms.

Axiom ($\mathrm{K}_\varnothing$) follows from axiom ($\mathrm{F}_{\mathrm{Proper}}$) as $[\![\exists h \in \varnothing : \Pr(h) > 0]\!] = \varnothing \notin \mathcal{F}$.

Axiom ($\mathrm{K}_{\setminus\{0\}}$): Note that $P(0) = 0$. So for every p, if $g \in A$ with $P(g) > 0$, then $g \in A \setminus \{0\}$; thus $[\![\exists h \in A : \Pr(h) > 0]\!] \subseteq [\![\exists h \in A \setminus \{0\} : \Pr(h) > 0]\!]$; so $[\![\exists h \in A : \Pr(h) > 0]\!] \in \mathcal{F}$ implies $[\![\exists h \in A \setminus \{0\} : \Pr(h) > 0]\!] \in \mathcal{F}$ by axiom ($\mathrm{F}_\supseteq$), as required.

Axiom ($\mathrm{K}_\supseteq$) follows from axiom ($\mathrm{F}_\supseteq$) because $B \supseteq A$ implies $[\![\exists h \in B : \Pr(h) > 0]\!] \supseteq [\![\exists h \in A : \Pr(h) > 0]\!]$.

Axiom ($\mathrm{K}_\geq$): observe that if $P(g) > 0$ then $P(f_g) > 0$, so it holds by axiom ($\mathrm{F}_\supseteq$).

Axiom ($\mathrm{K}_{\mathrm{scalar}}$): observe that if $P(g) > 0$ then $P(\lambda_g g) > 0$, so $P \in [\![\exists h \in A : \Pr(h) > 0]\!] \implies P \in [\![\exists h \in \{\lambda_g g \mid g \in A\} : \Pr(h) > 0]\!]$. Thus axiom ($\mathrm{K}_{\mathrm{scalar}}$) holds by axiom ($\mathrm{F}_\supseteq$).

Axiom ($\mathrm{K}_{\mathrm{conv}}$): Let $B = \{f_{\langle g_1,\ldots,g_n\rangle} | \langle g_1, \ldots, g_n\rangle \in A_1, \ldots, A_n\}$ with $f_{\langle g_1,\ldots,g_n\rangle} \in \mathsf{posi}(\{g_1, \ldots, g_n\})$.

First observe that if $f_{\langle g_1,\ldots,g_n\rangle} \in \mathsf{posi}(\{g_1, \ldots, g_n\})$ and P is a linear prevision with $P(g_i) > 0$ for all g_i, then also $P(f_{\langle g_1,\ldots,g_n\rangle}) > 0$ by their linearity: $P(f_{\langle g_1,\ldots,g_n\rangle}) = P(\sum_i \lambda_i g_i) = \sum_i \lambda_i P(g_i) > 0$.

If $P^* \in [\![\exists h \in A_1 : \Pr(h) > 0]\!] \cap \ldots \cap [\![\exists h \in A_n : \Pr(h) > 0]\!]$ then there is some $g_1^* \in A_1, \ldots, g_n^* \in A_n$ with $P^*(g_i^*) > 0$ for each i. Thus, $P^*(f_{\langle g_1^*,\ldots,g_n^*\rangle}) > 0$. Thus $P^* \in [\![\exists h \in B : \Pr(h) > 0]\!]$.

We have thus shown that $[\![\exists h \in A_1 : \Pr(h) > 0]\!] \cap \ldots \cap [\![\exists h \in A_n : \Pr(h) > 0]\!] \subseteq [\![\exists h \in B : \Pr(h) > 0]\!]$

So if $A_1, \ldots, A_n \in \mathcal{K}_\mathcal{F}$, then $[\![\exists h \in A_1 : \Pr(h) > 0]\!], \ldots, [\![\exists h \in A_n : \Pr(h) > 0]\!] \in \mathcal{F}$, and thus $[\![\exists h \in B : \Pr(h) > 0]\!] \in \mathcal{F}$ by Proposition 2.7, giving us $B \in \mathcal{K}_\mathcal{F}$.

Axiom ($\mathrm{K}_{\gneq 0}$): Suppose $f \in \mathcal{G}_{\gneq 0}$. Consider some ω^* such that $f(\omega^*) > 0$. If p is a probability with $p(\omega^*) > 0$, then $P(f) > 0$. By axiom ($\mathrm{F}_{\text{reg-pw}}$), $[\![\mathrm{pr}(\omega^*) > 0]\!] \in \mathcal{F}$, and therefore $[\![\exists h \in \{f\} : \Pr(h) > 0]\!] \supseteq [\![\mathrm{pr}(\omega^*) > 0]\!]$ is also in $\mathcal{F}$ by axiom ($\mathrm{F}_\supseteq$).

□

For this is true, it is important that we have not extended given an infinite combination axiom, as De Cooman et al. [16] do in their K_5, calling our axiom system *finitely* coherent. For their notion of 'coherence' they include the following axiom:[17]

(K_{infconv}) If $A_i \in \mathcal{K}$ for each $i \in I$ (where I is possibly infinite) and for each sequence $\langle g_i \rangle_{i \in I}$ with $g_i \in A_i$ for each $i \in I$, $f_{\langle g_i \rangle_i}$ is some member of $\mathsf{conv}(\{g_i \mid i \in I\})$, then $\{f_{\langle g_i \rangle_i} \mid \langle g_i \rangle_i \in \bigtimes_{i \in I} A_i\} \in \mathcal{K}$

This is also adopted in De Bock [11]. But it is not satisfied in every $\mathcal{K}_{\mathcal{F}}$.

Proposition 4.4. *Coherent and pointwise regular $\mathcal{F}$ may result in a set of desirable gamble sets, $\mathcal{K}_{\mathcal{F}}$, which violate axiom* ($K_{\text{infconv}}$).

Proof. Fix $\Omega = \{\text{H}, \text{T}\}$ and consider $\mathcal{F}_{\text{InfBiased}}$ as in Example 2.6 (see also Example 3.12); that is $[\![\Pr(\text{H}) > 0.5]\!] \in \mathcal{F}_{\text{InfBiased}}$ and also $[\![\Pr(H) < 0.5 + \epsilon]\!] \in \mathcal{F}_{\text{InfBiased}}$ for every positive real number ϵ. Consider the gambles as in Example 3.12

$$a_\epsilon := \langle -0.5, 0.5 \rangle + \langle \epsilon, \epsilon \rangle = \langle -0.5 + \epsilon, 0.5 + \epsilon \rangle \text{ for } \epsilon \text{ any positive real.}$$

So, for example $a_{0.001} = \langle -0.499, 0.501 \rangle$.

Observe that $P(a_\epsilon) > 0$ iff $P(\text{H}) < 0.5 + \epsilon$. Thus, since $[\![\Pr(H) < 0.5 + \epsilon]\!] \in \mathcal{F}_{\text{InfBiased}}$ for each positive real ϵ, each singleton $\{a_\epsilon\}$ is in $\mathcal{K}_{\mathcal{F}_{\text{InfBiased}}}$.

Consider also

$$A^* := \{-a_\epsilon \mid \epsilon \text{ a positive real}\}$$

Consider any linear prevision, or probability, P with $P(\text{H}) > 0.5$. Since they are real-valued, there must be some positive real ϵ with $P(\text{H}) > 0.5 + \epsilon$. For this ϵ, then, $P(-a_\epsilon) > 0$. Therefore $[\![\exists h \in A^* : \Pr(h) > 0]\!] \supseteq [\![\Pr(\text{H}) > 0.5]\!]$. Thus, since $[\![\Pr(\text{H}) > 0.5]\!] \in \mathcal{F}_{\text{InfBiased}}$, using Definition 4.1, we have that $A^* \in \mathcal{K}_{\mathcal{F}_{\text{InfBiased}}}$.

We can now show that axiom (K_{infconv}) fails for $\mathcal{K}_{\mathcal{F}_{\text{InfBiased}}}$. Consider the infinitely many sets $A_1 = A^*$, $A_2 = \{a_{1/2}\}$, $A_3 = \{a_{1/3}\}$, and so on. We have observed that each is a member of $\mathcal{K}_{\mathcal{F}_{\text{InfBiased}}}$.

To evaluate axiom (K_{infconv}), we need to work with sequences $\langle g_i \rangle$ where $g_i \in A_i$ for each i. Such sequences must have the form $\langle -a_\epsilon, a_{1/2}, a_{1/3}, \ldots \rangle$ where ϵ is a positive real. For each ϵ, when $n \geq 1/\epsilon$ then $\epsilon \geq 1/n$ so $-a_\epsilon + a_{1/n} \leq 0$. Put $f_\epsilon = -a_\epsilon + a_{1/N_\epsilon}$ for some $N_\epsilon \geq 1/\epsilon$. Axiom ($K_{\text{infconv}}$) would require that $\{f_\epsilon \mid \epsilon > 0\} \in \mathcal{K}_{\mathcal{F}_{\text{InfBiased}}}$. But since $f_\epsilon \leq 0$, $P(f_\epsilon) \leq 0$ for all P; thus $[\![\exists h \in \{f_\epsilon \mid \epsilon > 0\} : \Pr(h) > 0]\!] = \varnothing \notin \mathcal{F}$. □

It is also important that the Archimedean axiom of De Bock and de Cooman [13, §9] is not imposed [see also 53, AA3]. More generally, it is important that we have not added any further axioms which are not derivable from the axioms in Definition 4.2 along with that of Definition 4.5. This is a consequence of our main result in the next section, Theorem 4.8, which shows that every set of desirable gamble sets satisfying these axioms is obtained from some probability filter. So any axiom that goes beyond the specified axioms must fail for some $\mathcal{K}_{\mathcal{F}}$.

[17] They actually give a more general version of it, involving the 'closure' notion in general. It certainly entails this axiom (K_{infconv}).

4.3 The mixing axiom

However, using these axioms, we obtain a negative answer to our second question: some coherent $\mathcal{K}$ cannot be represented as $\mathcal{K}_{\mathcal{F}}$. This is because of our choice to associate a filter with a choice function using the analogue of E-admissibility.

There is a further axiom which is considered in De Bock and de Cooman [13, §8] which restricts to those choice functions determined in accordance with E-admissibility.[18]

This is not an axiom which they endorse. Following Van Camp [60], they will also allow as coherent those choice functions which are derived from a credal set in accordance with Maximality, as well as a whole range of yet unconsidered choice functions which are coherent.

Since we are deriving a choice function from a probability filter by a generalisation of E-admissibility, it is a principle that we obtain. To consider also representation of the non-mixing choice functions, De Cooman et al. [16] consider a more general framework, which we will discuss in Section 4.4.

The mixing axiom they give is closely related to Seidenfeld et al. [53, Axiom 2b]. We give the axiom in an analogous way to De Bock and de Cooman [13, §8] but also consider the *closure* of the convex hull of a set, as Seidenfeld et al. [53] do. This is not relevant to De Bock and de Cooman [13] as they restrict attention to finite sets of gambles.

($\mathrm{K}_{\mathrm{Mix}}$) If $A \in \mathcal{K}$ and $\mathsf{clconv}(B) \supseteq A \supseteq B$ then $B \in \mathcal{K}$.

In fact, we will impose the two components of the mixing axiom separately for perspicuity, allowing a reader to determine where the different components are playing a role in the proofs.

Definition 4.5. $\mathcal{K}$ is **mixing** if it satisfies:

($\mathrm{K}_{\text{Mix-conv}}$) If $A \in \mathcal{K}$ and $\mathsf{conv}(B) \supseteq A \supseteq B$ then $B \in \mathcal{K}$.

($\mathrm{K}_{\text{Mix-cl}}$) If $A \in \mathcal{K}$ and $\mathsf{closure}(B) \supseteq A \supseteq B$ then $B \in \mathcal{K}$.

This is the topological closure in the sup-norm topology, or topology of uniform convergence. That is, if $f_n \in B$ and f_n uniformly converge to f^*, $\sup_{\omega\in\Omega} |f_n(\omega) - f^*(\omega)| \longrightarrow 0$ as $n \longrightarrow \infty$, then $f^* \in B$.

Consider $\{g + \epsilon, -g + \epsilon\}$. Every probability evaluates one of these to have positive expected value, however different probability functions disagree on which. It is thus always deemed a desirable gamble set in accordance with Definition 4.1.

Theorem 4.6. *If $\mathcal{F}$ is coherent and pointwise regular, then $\mathcal{K}_{\mathcal{F}}$ is mixing coherent.*

Proof. In Proposition 4.3 we have already shown that $\mathcal{K}_{\mathcal{F}}$ is coherent. We now need to show that it is mixing.

Suppose $A \in \mathcal{K}_{\mathcal{F}}$, i.e., $[\![\exists h \in A : \Pr(h) > 0]\!] \in \mathcal{F}$, and suppose also that $\mathsf{conv}(B) \supseteq A \supseteq B$. We need to show that $[\![\exists h \in B : \Pr(h) > 0]\!] \in \mathcal{F}$. By axiom ($\mathrm{F}_{\supseteq}$), it suffices to show that $[\![\exists h \in A : \Pr(h) > 0]\!] \subseteq [\![\exists h \in B : \Pr(h) > 0]\!]$. Suppose $P \in [\![\exists h \in A : \Pr(h) > 0]\!]$, so there is some $f^* \in A$ with $P(f^*) > 0$.

[18] For the connection of their paper they are considering finite gamble sets and also impose an Archimedean axiom and show representation with a credal set.

$f^* \in A \subseteq \mathsf{conv}(B)$, so there are $g_1, \ldots, g_n \in B$ and $\lambda_1, \ldots, \lambda_n > 0$ with $f^* = \sum_i \lambda_i g_i$. By linearity, $0 < P(f^*) = \sum_i \lambda_i P(g_i)$, so for at least one of these $g_i \in B$, $P(g_i) > 0$. Thus $P \in [\![\exists h \in B : \Pr(h) > 0]\!]$. So $[\![\exists h \in A : \Pr(h) > 0]\!] \subseteq [\![\exists h \in B : \Pr(h) > 0]\!]$, as required.

Now suppose that $A \in \mathcal{K}_{\mathcal{F}}$ and $\mathsf{closure}(B) \supseteq A \supseteq B$. Again we just need to show that $[\![\exists h \in A : \Pr(h) > 0]\!] \subseteq [\![\exists h \in B : \Pr(h) > 0]\!]$. Suppose $P \in [\![\exists h \in A : \Pr(h) > 0]\!]$, so there is some $f^* \in A$ with $P(f^*) > 0$. $f^* \in A \subseteq \mathsf{closure}(B)$, so $f^* \in \mathsf{closure}(B)$. Therefore, there is some sequence $\langle h_n \rangle$ with each $h_n \in B$ and which converges uniformly to f^*. Since linear previsions are continuous with respect to uniform convergence, $P(h_N) > 0$ for some N.[19] Since $h_N \in B$, thus $P \in [\![\exists h \in B : \Pr(h) > 0]\!]$. We have thus shown that $[\![\exists h \in A : \Pr(h) > 0]\!] \subseteq [\![\exists h \in B : \Pr(h) > 0]\!]$, as required. □

Before moving to question (ii) and showing that every mixing coherent $\mathcal{K}$ is obtained from some coherent $\mathcal{F}$, we will make a detour to discuss an alternative model, that of filters of coherent sets of desirable gambles as developed by De Cooman et al. [16].

4.4 Filters of coherent sets of desirable gambles

In [16, Theorem 25], De Cooman et al. show that all coherent choice functions (what they call *finitely* coherent choice functions) can be recovered as filters not of probabilities (or linear previsions), as we do, but directly as filters of coherent sets of desirable gambles.[20]

Instead of focusing on constraints on probabilities or linear previsions, their filters encode constraints on coherent sets of desirable gambles. For example, it might contain $\{\mathcal{D}$ a coherent set of desirable gambles $\mid g \in \mathcal{D}\}$ for a particular gamble g. The 'points' in the filters are changed from probability functions (or linear previsions) to coherent sets of desirable gambles.

This offers a more general framework than ours since distinct probability functions determine distinct sets of desirable gambles (the Archimedean and mixing ones). The representational power of probability filters is then exactly that of filters of Archimedean and mixing coherent sets of desirable gambles. By allowing also for other coherent sets of desirable gambles they obtain a more general framework which can capture all coherent choice functions.

De Cooman et al. identify the inference mechanism behind all coherent sets of desirable gamble sets in terms of (finitary) entailment according to sets of desirable gambles (that is, binary judgements of comparative expectation). We identify the inference mechanism of the coherent mixing ones by finitary probabilistic entailment amongst credal judgements of gamble sets as having something with positive expected value.

Many defenders of the credal set model will wish to reject all coherent choice functions as legitimate, for example Levi [37] argues for E-admissibility, in which

[19] More carefully: let $\epsilon := P(f^*)/2 > 0$. Since h_n are assumed to converge uniformly to f^*, then there is some N such that $\|h_N - f\|_\infty < \epsilon$. Thus $\inf\{h_N(\omega) - f^*(\omega) \mid \omega \in \Omega\} \geq -\epsilon$, so $P(h_N - f^*) \geq -\epsilon$. Thus $P(h_N) = P(f^*) + P(h_N - f^*) > \epsilon + -\epsilon = 0$.

[20] This extends the results of [11, 13] showing that choice functions satisfying axiom ($\mathrm{K}_{\mathrm{infconv}}$) or those restricted to finite gamble sets respectively are recovered from sets of coherent sets of desirable gambles, or equivalently, the principal filters of $\mathcal{D}$s, in the same way that credal sets are equivalent to principal probability filters.

case the more restricted representational power of the probability filter model is acceptable.

Alternatively, to allow for all coherent choice functions, we suggest that one could use probability filters as the model of the belief state of the agent and include an explicit parameter describing how she structures her choices, or reduces judgements of a gamble set being desirable to matters of binary comparisons.[21] This parallels the idea of having a credal set and making a decision in accordance with a choice rule such as E-admissibility or Maximality. One then offers rationalising explanations for an agent's choice behaviour in terms of her beliefs, taking the form of a probability filter, plus her desires and her approach to structuring choices.[22] Specifying the form of this parameter in a way which allows all coherent choice functions to be derived is a topic of future investigation.

This is not merely a matter of relabelling components of the choice function into a 'belief' component, having the form of a probability filter, and a 'choice-mechanism' component. If we assume a certain amount of variation in these parameters as legitimate, they cannot be recovered from knowing the agent's choice function. Distinct credal sets may result in different choice functions in accordance with E-admissibility whilst nonetheless resulting in the same choice function under Maximality. So one's choice function does not include enough information to determine one's belief state and choice mechanism. One might argue that this is the right diagnosis: that there are no such differences of opinion without a difference in decision making recommendations, i.e., in associated choice functions. But a more epistemic approach to belief might consider these distinctions to be important, as is prominent for example in Joyce [28]. This also results in the slightly odd result that which differences in credal set are representationally significant depends on one's personal approach to choice, even though different approaches are acceptable. What this indicates is that it requires a full-blown replacement of the idea of credal sets as a model of belief.

An advantage of the probability filter model is its close connection with the prominent credal set models. We are arguing in this paper that it is a modification of that model which allows it to avoid the challenges to the credal set model given by Walley [68].

The results of this paper also have value without accepting the probability filters as a model of belief. One way to view our results is to provide an independent notion of the logic of desirability via the notion of finitary probabilistic entailment amongst judgements of gambles having positive expected value. It is then of significant interest that it matches usual axioms of coherence on desirable gambles. Whilst the analogue results become much simpler in the formalism of filters of coherent sets of desirable gambles, requiring no hyperplane separation result, they do not show this feature. Our results are valuable in that they highlight a tight connection between axioms of desirability and the laws of probability.

[21]See also Buchak [4, p.53-54] for an explicit discussion of including an further parameter which plays the role in structuring choices.

[22]Joyce [25, §1.3] argues that it is very important to provide such rationalizing explanations for choice behaviour. He uses this as a criticism of behaviourism.

4.5 Question (ii). Every mixing coherent $\mathcal{K}$ is obtained from some $\mathcal{F}$.

We will now show that by including the mixing axiom, all mixing coherent sets of desirable gamble sets are obtained from a probability filter. This shows us that when we understand a judgement of a gamble set as desirable in line with Definition 4.1, these axioms correspond to such judgements being consistent and closed under finitary probabilistic entailment, along with the assumption that each possible outcome has positive probability to get axiom ($\mathrm{K}_{\gneq 0}$).

Definition 4.7. For a set of sets of gambles, $\mathcal{K} \subseteq \wp(\mathcal{G})$, define $\mathcal{F}_{\mathcal{K}}$ by:

$$\mathcal{F}_{\mathcal{K}} := \mathrm{ext}(\{[\![\exists h \in A : \Pr(h) > 0]\!] \mid A \in \mathcal{K}\})$$

Recalling the definition of ext, we have $C \in \mathcal{F}_{\mathcal{K}}$ iff there are some finitely many gamble sets, $A_1, \ldots, A_n \in \mathcal{K}$ with

$$C \supseteq [\![\exists h \in A_1 : \Pr(h) > 0]\!] \cap \ldots \cap [\![\exists h \in A_n : \Pr(h) > 0]\!]$$

That is,

$$C \supseteq \left\{ P \in \mathrm{Prevs} \;\middle|\; \begin{array}{l} \exists h_1 \in A_1 \text{ s.t. } P(h_1) > 0 \\ \text{and } \ldots \\ \text{and } \exists h_n \in A_n \text{ s.t. } P(h_n) > 0 \end{array} \right\}.$$

Theorem 4.8. *For any mixing coherent set of desirable gamble sets, $\mathcal{K}$, $\mathcal{F}_{\mathcal{K}}$ is coherent and pointwise regular and $\mathcal{K} = \mathcal{K}_{\mathcal{F}_{\mathcal{K}}}$, i.e.,*

$$B \in \mathcal{K} \text{ iff } [\![\exists h \in B : \Pr(h) > 0]\!] \in \mathcal{F}_{\mathcal{K}} \tag{3}$$

Let

$$S_B := \mathsf{cone}(B) - \mathcal{G}_{\geq 0}.$$

This extends the notion S_b as used in Theorem 3.10. Our proof is closely related to that of Theorem 3.10.

Sublemma 4.8.1.

$$S_B = (\mathsf{posi}(B) - \mathcal{G}_{\geq 0}) \;\cup\; \mathcal{G}_{\leq 0}.$$

Proof.

$$\begin{aligned} S_b &:= \mathsf{cone}(B) \;-\; \mathcal{G}_{\geq 0} \\ &= (\mathsf{posi}(B) \cup \{0\}) \;-\; \mathcal{G}_{\geq 0} \\ &= (\mathsf{posi}(B) - \mathcal{G}_{\geq 0}) \;\cup\; (\{0\} - \mathcal{G}_{\geq 0}) \\ &= (\mathsf{posi}(B) - \mathcal{G}_{\geq 0}) \;\cup\; \mathcal{G}_{\leq 0}. \end{aligned}$$ □

Sublemma 4.8.2. *Suppose $\mathcal{K}$ is coherent and mixing. If $\mathsf{closure}(S_B) \in \mathcal{K}$ then $B \in \mathcal{K}$.*

Proof. By Sublemma 4.8.1, $S_B \setminus \mathcal{G}_{\leq 0} \subseteq \mathsf{posi}(B) - \mathcal{G}_{\geq 0}$

We can then argue as follows:

$$\begin{aligned} \mathsf{closure}(S_B) \in \mathcal{K} &\implies S_B \in \mathcal{K} && \text{axiom } (\mathrm{K}_{\text{Mix-cl}}) \\ &\implies S_B \setminus \mathcal{G}_{\leq 0} \in \mathcal{K} && \text{axioms } (\mathrm{K}_{\setminus\{0\}}) \text{ and } (\mathrm{K}_{\geq}) \\ &\implies \mathsf{posi}(B) - \mathcal{G}_{\geq 0} \in \mathcal{K} && \text{axiom } (\mathrm{K}_{\supseteq}) \\ &\implies \mathsf{posi}(B) \in \mathcal{K} && \text{axiom } (\mathrm{K}_{\geq}) \\ &\implies \mathsf{conv}(B) \in \mathcal{K} && \text{axiom } (\mathrm{K}_{\text{scalar}}) \\ &\implies B \in \mathcal{K} && \text{axiom } (\mathrm{K}_{\text{Mix-conv}}) \end{aligned}$$ □

We now give a result which extends Sublemma 3.10.3. It is illustrated in Fig. 4.

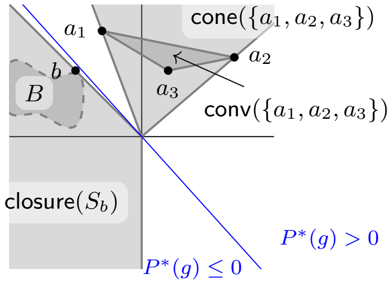

Figure 4: Illustrates the proof of Sublemma 4.8.3. When $\mathsf{conv}(\{a_1, a_2, a_3\})$ and $\mathsf{closure}(S_B)$ are disjoint, then there is a linear prevision P^* with $P^*(a_i) > 0$ for each i but $P^*(b) \leq 0$ for all $b \in B$.

Sublemma 4.8.3. *If* $\mathsf{conv}(\{a_1, \dots, a_n\})$ *and* $\mathsf{closure}(S_B)$ *are disjoint, then there is a linear prevision* P^* *with* $P^*(a_i) > 0$ *for all* i *but* $P^*(b) \leq 0$ *for all* $b \in B$*.*

Proof. The proof is exactly as in Sublemma 3.10.3. We briefly repeat the argument. Observe that S_B is a convex cone, so $\mathsf{closure}(S_B)$ is a closed convex cone. $\mathsf{cone}(\{a_1, \dots, a_n\})$ is a locally compact closed convex cone (since it is finitely generated). By assumption that $\mathsf{conv}(\{a_1, \dots, a_n\})$ and $\mathsf{closure}(S_B)$ are disjoint, $\mathsf{cone}(\{a_1, \dots, a_n\})$ and $\mathsf{closure}(S_B)$ intersect only at 0. So by Theorem 2.5 of Klee [30], there is a continuous linear functional T with $T(a_i) > 0$ for each i, and $T(g) \leq 0$ for all $g \in \mathsf{closure}(S_B)$. It can be checked that it is positive and thus monotone. To show it can be normalised we show that $T(\mathbf{1}_\Omega) > 0$. Since a_1 is bounded, there is some $\lambda > 0$, such that $\lambda a_1 \leq \mathbf{1}_\Omega$; and so, by monotonicity, $T(\mathbf{1}_\Omega) \geq T(\lambda a_1)$. By construction, $T(a_1) > 0$, so $T(\mathbf{1}_\Omega) \geq T(\lambda a_1) = \lambda T(a_1) > 0$. It can thus be normalised and obtains our required linear prevision. □

Proof of Theorem 4.8. Assume $\mathcal{K}$ is coherent. We will first show that $\mathcal{K} = \mathcal{K}_{\mathcal{F}_\mathcal{K}}$, i.e, Eq. (3), before checking that $\mathcal{F}_\mathcal{K}$ is coherent and pointwise regular.

For any $B \in \mathcal{K}$ we have $[\![\exists h \in B : \Pr(h) > 0]\!] \in \mathcal{F}_\mathcal{K}$ by construction. We need to show the converse, that is, if $[\![\exists h \in B : \Pr(h) > 0]\!] \in \mathcal{F}_\mathcal{K}$ then $B \in \mathcal{K}$.

By definition of $\mathcal{F}_\mathcal{K}$, if $[\![\exists h \in B : \Pr(h) > 0]\!] \in \mathcal{F}_\mathcal{K}$ then there are some $A_1, \dots, A_n \in \mathcal{K}$ where $[\![\exists h \in B : \Pr(h) > 0]\!] \supseteq [\![\exists h \in A_1 : \Pr(h) > 0]\!] \cap \dots \cap [\![\exists h \in A_n : \Pr(h) > 0]\!]$. We need to show that this entails that $B \in \mathcal{K}$.

If $\langle a_1, \dots, a_n \rangle \in A_1 \times \dots \times A_n$ with $\mathsf{conv}(\{a_1, \dots, a_n\})$ and $\mathsf{closure}(S_B)$ disjoint, then by Sublemma 4.8.3, there is some linear prevision with $P^*(a_i) > 0$ for all $a_1, \dots, a_n$ and $P^*(b) \leq 0$ for all $b \in B$. Such P^* would then be in each $[\![\exists h \in A_i : \Pr(h) > 0]\!]$ but not in $[\![\exists h \in B : \Pr(h) > 0]\!]$, violating our superset claim.

Thus, for every sequence $\langle a_1, \dots, a_n \rangle \in A_1 \times \dots \times A_n$, $\mathsf{conv}(\{a_1, \dots, a_n\})$ and $\mathsf{closure}(S_B)$ are non-disjoint. By axioms ($\mathrm{K}_\supseteq$) and ($\mathrm{K}_{\mathrm{conv}}$), thus $\mathsf{closure}(S_B) \in \mathcal{K}$. Therefore $B \in \mathcal{K}$ by Sublemma 4.8.2.

We have thus shown that if $\mathcal{K}$ is mixing coherent and there are some $A_1, \dots, A_n \in \mathcal{K}$ where $[\![\exists h \in B : \Pr(h) > 0]\!] \supseteq [\![\exists h \in A_1 : \Pr(h) > 0]\!] \cap \dots \cap$

$[\![\exists h \in A_n : \Pr(h) > 0]\!]$ then $B \in \mathcal{K}$, as required. This gives us that $\mathcal{K} = \mathcal{K}_{\mathcal{F}_{\mathcal{K}}}$, i.e., Eq. (3).

It remains to show that $\mathcal{F}_{\mathcal{K}}$ is coherent and pointwise regular.

By definition of ext, $\mathcal{F}_{\mathcal{K}}$ satisfies axioms ($F_{\cap}$) and ($F_{\supseteq}$). By axiom ($K_{\gneqq 0}$), $K \neq \varnothing$, so $\mathcal{F}_{\mathcal{K}}$ satisfies axiom ($F_{\neq\varnothing}$). To observe that $\mathcal{F}_{\mathcal{K}}$ satisfies axiom (F_{Proper}), observe that $\varnothing = [\![\exists h \in \varnothing : \Pr(h) > 0]\!] \notin \mathcal{F}_{\mathcal{K}}$, by our proved Eq. (3) and by axiom ($K_{\varnothing}$), $\varnothing \notin \mathcal{K}$.

For axiom ($F_{\text{reg-pw}}$): Note that $[\![\exists h \in \{\mathbf{1}_{\omega^*}\} : \Pr(h) > 0]\!] = [\![\mathrm{pr}(\omega^*) > 0]\!]$. By axiom ($K_{\gneqq 0}$), $\{\mathbf{1}_{\omega^*}\} \in \mathcal{K}$, thus $[\![\mathrm{pr}(\omega^*) > 0]\!] \in \mathcal{F}_{\mathcal{K}}$ by our proved Eq. (3). □

Corollary 4.9. *For distinct mixing coherent $\mathcal{K}$ and $\mathcal{K}'$, $\mathcal{F}_{\mathcal{K}}$ and $\mathcal{F}_{\mathcal{K}'}$ are distinct.*

Proof. This follows immediately from Theorem 4.8. □

4.6 Without Regularity

Before moving to discuss question (iii), we discuss what can be maintained of our results if regularity is dropped. This then tells us exactly what corresponds to finite probabilistic entailment when the $A \in \mathcal{K}$ is understood as a judgement given by $[\![\exists h \in A : \Pr(h) > 0]\!]$.

Definition 4.10. $\mathcal{K}$ is **irregularly coherent** replacing axiom ($K_{\gneqq 0}$) of Definition 4.2 with:

($K_{\inf>0}$) If $\inf(g) > 0$, then $\{g\} \in \mathcal{K}$

Unlike the case of desirable gambles, the proof immediately applies also to the irregular case without modification or extra axioms to be imposed, so we directly present it in that form.

Theorem 4.11. *If $\mathcal{F}$ is coherent, then $\mathcal{K}_{\mathcal{F}}$ is mixing irregularly coherent.*

Proof. In the proof of Theorem 4.6, pointwise regularity was only used in deriving axiom ($K_{\gneqq 0}$).

Axiom ($K_{\inf>0}$): If $\inf(g) > 0$, then every $P \in \text{Prevs}$ has $P(g) \geq \inf(g) > 0$, so $[\![\exists h \in \{g\} : \Pr(h) > 0]\!] = \text{Prevs} \in \mathcal{F}$ (axiom ($F_{\neq\varnothing}$) and ($F_{\supseteq}$)).

□

Theorem 4.12. *If $\mathcal{K}$ is mixing irregularly coherent, then $\mathcal{F}_{\mathcal{K}}$ is coherent and $\mathcal{K} = \mathcal{K}_{\mathcal{F}_{\mathcal{K}}}$.*

Proof. The proof of Theorem 4.8 does not make use of axiom ($K_{\gneqq 0}$) except to ensure that $\mathcal{K}$ is non-empty, which we instead obtain from axiom ($K_{\inf>0}$), and in showing that $\mathcal{F}_{\mathcal{K}}$ is pointwise regular, which is not needed for this result. □

This works even when Ω is infinite, unlike the desirable gambles case where we needed to add axiom (D_{ArchCl}). There seems to be a close relationship between the closure component of the mixing axiom and axiom (D_{ArchCl}), although the exact relationship needs investigating. Note that it is not the relationship as discussed in De Cooman et al. [16] because here the $\mathcal{K}$ are not being obtained as filters of these $\mathcal{D}$ satisfying axiom (D_{ArchCl}) but rather both $\mathcal{D}$ and $\mathcal{K}$ are being obtained from filters of linear previsions, or Archimedean mixing $\mathcal{D}$.

4.7 Question (iii). Probability filters go beyond desirability of gamble sets.

An encoding of which gamble sets are desirable by Definition 3.4, does not suffice to tell us everything about the opinion state, as given by a probability filter. This contrasts to the case of credal sets, where Archimedean mixing coherent choice functions are expressively equivalent to sets of probability functions [53]. The expressive power of probability filters goes strictly beyond that of sets of desirable gamble sets.

Theorem 4.13. *There are distinct coherent probability filters, $\mathcal{F}$ and $\mathcal{F}'$ which result in the same sets of desirable gamble sets, i.e., $\mathcal{K}_{\mathcal{F}} = \mathcal{K}_{\mathcal{F}'}$.*

We couch the proof of this by setting up an example, which bears a close relationship to $\mathcal{F}_{\text{InfBiased}}$ used in Examples 2.6 and 3.12, and as was used in the proof of Lemma 4.15.

Example 4.14. We will describe two filters, $\mathcal{F}_{\text{Fair}}$ and $\mathcal{F}_{\text{FairOrInfBiased}}$.
Consider $\Omega = \{\text{H}, \text{T}\}$, the outcomes of a coin toss. Define:

$$\mathcal{F}_{\text{Fair}} := \{C \subseteq \text{Prevs} \mid C \supseteq [\![\text{pr(H)} = 0.5]\!]\}$$

$$\mathcal{F}_{\text{FairOrInfBiased}} := \left\{ C \subseteq \text{Prevs} \;\middle|\; \begin{array}{l} \text{there is some } \epsilon \in \mathbb{R} \text{ with } \epsilon > 0 \text{ and} \\ C \supseteq [\![0.5 \leq \text{pr(H)} < 0.5 + \epsilon]\!] \end{array} \right\}.$$

In a similar way to Example 2.6, one can check that these are coherent. They are also pointwise regular.

Unlike for $\mathcal{F}_{\text{Fair}}$, for $\mathcal{F}_{\text{FairOrInfBiased}}$ it does not seem equally likely that the coin lands heads as tails. She suspends judgement on whether it is equally likely or more likely to land heads than tails.

$$[\![\text{pr(H)} = 0.5]\!] \notin \mathcal{F}_{\text{FairOrInfBiased}}$$

So $\mathcal{F}_{\text{Fair}}$ and $\mathcal{F}_{\text{FairOrInfBiased}}$ are distinct.

However, this difference between $\mathcal{F}_{\text{Fair}}$ and $\mathcal{F}_{\text{FairOrInfBiased}}$ does not affect any judgements of the (strict) desirability of gambles, or of whether gamble sets contain a desirable gamble. That is what we show in the next result.

Lemma 4.15. $\mathcal{K}_{\mathcal{F}_{\text{Fair}}} = \mathcal{K}_{\mathcal{F}_{\text{FairOrInfBiased}}}$

Proof. Recall $A \in \mathcal{K}_{\mathcal{F}}$ iff $[\![\exists h \in A : \text{Pr}(h) > 0]\!] \in \mathcal{F}$.

Since $\mathcal{F}_{\text{FairOrInfBiased}} \subseteq \mathcal{F}_{\text{Fair}}$, also $\mathcal{K}_{\mathcal{F}_{\text{FairOrInfBiased}}} \subseteq \mathcal{K}_{\mathcal{F}_{\text{Fair}}}$. We need to show the converse.

Suppose $A \in \mathcal{K}_{\mathcal{F}_{\text{Fair}}}$, that is $[\![\exists h \in A : \text{Pr}(h) > 0]\!] \supseteq [\![\text{pr(H)} = 0.5]\!]$.

Take any $P^* \in [\![\text{pr(H)} = 0.5]\!]$. Since $P^* \in [\![\exists h \in A : \text{Pr}(h) > 0]\!]$ there is a $g^* \in A$ with $P^*(g^*) > 0$. Let $\delta > 0$ such that $P^*(g^*) > \delta > 0$.

Recall that this example is considering $\Omega = \{\text{H}, \text{T}\}$, so that for $P \in \text{Prevs}$, $P(g) = P(\text{H})g(\text{H}) + P(\text{T})g(\text{T})$.

If $g^*(\text{H}) \geq g^*(\text{T})$ then for $P(\text{H}) \geq 0.5$, we have $P(g^*) \geq P^*(g^*) > 0$. In this case, $[\![\exists h \in A : \text{Pr}(h) > 0]\!] \supseteq [\![\text{pr(H)} \geq 0.5]\!]$ so $[\![\exists h \in A : \text{Pr}(h) > 0]\!] \supseteq [\![\text{Pr}(g^*) > 0]\!] \in \mathcal{F}_{\text{FairOrInfBiased}}$.

If $g^*(\text{H}) < g^*(\text{T})$ then put

$$\epsilon := \frac{\delta}{g^*(\text{T}) - g^*(\text{H})} > 0.$$

and observe that if $0.5 \leq P(\mathrm{H}) < 0.5 + \epsilon$,

$$\begin{aligned} P^*(g^*) - P(g^*) &= (P(\mathrm{H}) - 0.5)(g^*(\mathrm{T}) - g^*(\mathrm{H})) \\ &\leq \epsilon(g^*(\mathrm{T}) - g^*(\mathrm{H})) && \text{as } g^*(\mathrm{T}) > g^*(\mathrm{H}) \\ &= \delta \end{aligned}$$

And by assumption that $P^*(g^*) > \delta$, we can conclude that $P(g^*) > 0$. This shows then that $[\![\exists h \in A : \Pr(h) > 0]\!] \supseteq [\![\Pr(g^*) > 0]\!] \supseteq [\![0.5 \leq (()\mathrm{H}) < \epsilon]\!]$, for this choice of ϵ. This therefore gives us $[\![\exists h \in A : \Pr(h) > 0]\!] \in \mathcal{F}_{\mathrm{FairOrInfBiased}}$. □

As in the case of desirability (Section 3.6), one could obtain equivalent frameworks if we restrict the probability filter framework. In this case, we would restrict to those which satisfy axiom ($\mathrm{F}_{\mathcal{K}}$):

($\mathrm{F}_{\mathcal{K}}$) If $C \in \mathcal{F}$ then there are some finitely many sets of gambles $A_1, \ldots, A_n$ with $[\![\exists h \in A_i : \Pr(h) > 0]\!] \in \mathcal{F}$ for each A_i and

$$C \supseteq [\![\exists h \in A_1 : \Pr(h) > 0]\!] \cap \ldots \cap [\![\exists h \in A_n : \Pr(h) > 0]\!].$$

This axiom ensures that $\mathcal{F}$ is determined by its treatment of the probability constraints of the form $[\![\exists h \in A : \Pr(h) > 0]\!]$.

A restriction essentially equivalent to this is adopted in de Cooman et al. [16, p.12], although they implement it by restricting the kinds of probability constraints considered. See Section 2.4 for more general discussion about how to impose restrictions.

Proposition 4.16. *For every coherent mixing $\mathcal{K}$, $\mathcal{F}_{\mathcal{K}}$ satisfies axiom* ($\mathrm{F}_{\mathcal{K}}$)*. Also, if $\mathcal{F}$ is a probability filter satisfying axiom* ($\mathrm{F}_{\mathcal{K}}$) *then $\mathcal{F} = \mathcal{F}_{\mathcal{K}_{\mathcal{F}}}$.*

Proof. The argument is exactly as in Proposition 3.19. □

We do not propose ($\mathrm{F}_{\mathcal{K}}$) as an axiom which should be adopted when capturing an agent's belief state, as it rules out important differences of opinion. For example, it rules out $\mathcal{F}_{\mathrm{Fair}}$. The credal judgement that the coin is equally likely to land heads and tails is not something that matters for judgements of (strict) desirability of any gambles (due to Lemma 4.15; it is only $\mathcal{F}_{\mathrm{FairOrInfBiased}}$ which satisfies axiom ($\mathrm{F}_{\mathcal{K}}$)). However, I propose that this is an important kind of opinion that should be captured in the model of belief. Thus, adopting axiom ($\mathrm{F}_{\mathcal{K}}$) is, I propose, too strong, and our representation of uncertainty should go beyond sets of desirable gamble sets.

4.8 Updating and Conditionalisation

Building on Section 3.7 finally move to showing how our account of updating on probability filters corresponds to that on choice functions. Again we result in a tight relationship with one's judgements on the called-off gambles.

As in Van Camp [60], we define:

Definition 4.17. Let $\mathbf{1}_E A := \{\mathbf{1}_E g \mid g \in A\}$,

Proposition 4.18. *Suppose $\mathcal{F}$ is coherent and pointwise regular. Then $A \in \mathcal{K}_{\mathcal{F}|E}$ iff $\mathbf{1}_E A \in \mathcal{K}_{\mathcal{F}}$.*

This works similarly to the proof of Proposition 3.23.

Proof. We need to show that $[\![\exists h \in A : \ \Pr(h) > 0]\!] \in \mathcal{F}|E$ iff $[\![\exists h \in \mathbf{1}_E A : \ \Pr(h) > 0]\!] \in \mathcal{F}$.

From the definition of $\mathcal{F}|E$, we have that $[\![\exists h \in A : \ \Pr(h) > 0]\!] \in \mathcal{F}|E$ iff there is some $C \in \mathcal{F}$ with $[\![\exists h \in A : \ \Pr(h) > 0]\!] \supseteq \{P(\cdot \mid E) \mid P \in C \text{ and } P(\mathbf{1}_E) > 0\}$. That is, for every $P \in C$ with $P(\mathbf{1}_E) > 0$, there is some $h \in A$ with $P(h|E) > 0$. By Proposition 3.21 this is iff $P(\mathbf{1}_E h) > 0$. We need to show that there is some such C iff $[\![\exists h \in \mathbf{1}_E A : \ \Pr(h) > 0]\!] \in \mathcal{F}$.

If $[\![\exists h \in \mathbf{1}_E A : \ \Pr(h) > 0]\!] \in \mathcal{F}$, then this is such a C.

Conversely, if we have such a C then since we also have $[\![\Pr(\mathbf{1}_E) > 0]\!] \in \mathcal{F}$ by pointwise regularity, we have $C \cap [\![\Pr(\mathbf{1}_E) > 0]\!] \in \mathcal{F}$. If P is in this intersection then $P(\mathbf{1}_E h) > 0$ for some $h \in A$, and so $P \in [\![\exists h \in \mathbf{1}_E A : \ \Pr(h) > 0]\!]$. By axiom ($\mathrm{F}_\supseteq$), then $[\![\exists h \in \mathbf{1}_E A : \ \Pr(h) > 0]\!] \in \mathcal{F}$. □

5 Conclusion

We have proposed representing an agent's uncertain belief state by a collection of credal judgements which are closed and consistent under finitary probabilistic entailment. More formally, they are represented with a collection of probability constraints, or constraints on linear previsions, which form the mathematical structure of a filter, that is, which are closed under finitary intersection and supersets, matching that of probabilistic entailment, and which don't contain $\varnothing$, ensuring that when we take such a finitary probabilistic closure, they don't result in inconsistencies.

The model is closely related to the credal set model of belief, with special kinds of filters, namely principal filters, being equivalent to the credal set model. By allowing also for non-principal filters as coherent we are instead just closing an agent's credal judgements under finitary probabilistic entailment. We have shown that this allows the account to accommodate a version of regularity and to encompasses the model of sets of desirable gambles. This allows it to avoid the objections by Walley [68] to the credal set model of belief.

We also showed a close connection between probabilistic entailment and coherence of sets of desirable gambles. We identified the axioms on sets of desirable gambles which matches that of finitary probabilistic entailment, understanding a judgement that a gamble is desirable to be satisfied by the probability functions evaluating it to have positive expected value (Section 3.5). We showed that, at least when we also include pointwise regularity, the assumption that gambles which might lead to a gain and cannot lead to a loss are judged as having positive expected payout, this results exactly in the usual axioms for coherence of a set of desirable gambles (Theorems 3.7 and 3.10). When we are interested in choice functions, the association becomes more fraught. We have identified the axioms that correspond to finitary probabilistic entailment amongst judgements of the form $[\![\exists h \in A : \ \Pr(h) > 0]\!]$ (Theorem 4.12); and when pointwise regularity is added these are axioms of mixing (finitely) coherent choice functions (Theorems 4.6 and 4.8).

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A Appendix

A.1 Proof of Theorems 3.13 and 3.15

Proposition A.1. *If $\mathcal{F}$ is coherent then $\mathcal{D}_{\mathcal{F}}$ satisfies axioms* ($\mathrm{D}_{0\notin}$)*,* (D_{λ}) *and* (D_{+}) *and axiom* ($\mathrm{D}_{\inf>0}$)*,* ($\mathrm{D}_{\geq}$) *and* ($\mathrm{D}_{\mathrm{ArchCl}}$)*.*

Proof. The proof that it satisfies axioms ($\mathrm{D}_{0\notin}$), (D_{λ}) and (D_{+}) is exactly as in Theorem 3.7, which do not make use of pointwise regularity.

Axiom ($\mathrm{D}_{\inf>0}$): If $\inf(g) > 0$ then for every $\Pr \in$ Prevs, $\Pr(g) \geq \inf(g) > 0$. So $[\![\Pr(g) > 0]\!] = \text{Prevs} \in \mathcal{F}$ and thus $g \in \mathcal{D}_{\mathcal{F}}$.

Axiom ($\mathrm{D}_{\geq}$): Suppose $g \in \mathcal{D}_{\mathcal{F}}$ and $f \geq g$. Then $[\![\Pr(g) > 0]\!] \in \mathcal{F}$. But every $\Pr \in$ Prevs, $\Pr(f) \geq \Pr(g)$, and thus $[\![\Pr(f) > 0]\!] \supseteq [\![\Pr(g) > 0]\!]$, so by axiom ($\mathrm{F}_{\supseteq}$), $[\![\Pr(f) > 0]\!] \in \mathcal{F}$, so $f \in \mathcal{D}_{\mathcal{F}}$.

Axiom ($\mathrm{D}_{\mathrm{ArchCl}}$): Suppose $a \in \mathcal{D}$ and $b \in \mathsf{posi}(\{a - \epsilon\}) + \mathcal{G}_{\geq 0}$ for all $\epsilon > 0$. Consider some $\Pr \in$ Prevs such that $\Pr(a) > 0$. Take $\epsilon > 0$ such that $\Pr(a) > \epsilon$. $b \in \mathsf{posi}(\{a - \epsilon\}) + \mathcal{G}_{\geq 0}$ so there is $\lambda > 0$ and $h \in \mathcal{G}_{\geq 0}$ such that $b = \lambda(a - \epsilon) + h$. So using the properties of linear previsions, $\Pr(b) = \lambda(\Pr(a) - \epsilon) + \Pr(h)$ with $\Pr(h) \geq 0$ and $\Pr(a) > \epsilon$, so $\Pr(b) > 0$, as required. □

This proves Theorems 3.13 and 3.15.

A.2 Proof of Theorem 3.14

Sublemmas 3.10.1 and 3.10.3 still hold and will be used. We need to find appropriate modifications of Sublemmas 3.10.2 and 3.10.4

Sublemma A.1.1. *Suppose $\mathcal{D}$ satisfies axioms* ($\mathrm{D}_{0\notin}$)*,* (D_λ) *and* ($\mathrm{D}_\geq$)*. If $\mathcal{D} \cap S_b \neq \varnothing$, then $b \in \mathcal{D}$.*

Proof. Suppose $f \in \mathcal{D} \cap S_b$.

$f \notin \mathcal{G}_{\leq 0}$: If $f \in \mathcal{G}_{\leq 0}$ then $0 \geq f$ so by axiom ($\mathrm{D}_\geq$), $0 \in \mathcal{D}$, contradicting axiom ($\mathrm{D}_{0\notin}$).

Thus in fact $b \in \mathsf{posi}(\{f\}) + \mathcal{G}_{\geq 0}$ by Sublemma 3.10.1. So $b = \lambda f + h$ with $h \in \mathcal{G}_{\geq 0}$ and $\lambda > 0$. By axiom (D_λ), $\lambda f \in \mathcal{D}$. Then by axiom ($\mathrm{D}_\geq$), $b \in \mathcal{D}$. □

Sublemma A.1.2. *If Ω is finite, then S_b is closed.*

Proof. When Ω is finite, say $\Omega = \{\omega_1, \ldots, \omega_n\}$, then $\mathcal{G}_{\geq 0} = \mathsf{cone}(\{\mathbf{1}_{\omega_1}, \ldots, \mathbf{1}_{\omega_n}\})$.

And thus

$$\begin{aligned} S_b &= \mathsf{cone}(\{b\}) - \mathcal{G}_{\geq 0} \\ &= \mathsf{cone}(\{b\}) - \mathsf{cone}(\{\mathbf{1}_{\omega_1}, \ldots, \mathbf{1}_{\omega_n}\}) \\ &= \mathsf{cone}(\{b\}) + \mathsf{cone}(\{-\mathbf{1}_{\omega_1}, \ldots, -\mathbf{1}_{\omega_n}\}) \\ &= \mathsf{cone}(\{b, -\mathbf{1}_{\omega_1}, \ldots, -\mathbf{1}_{\omega_n}\}) \end{aligned}$$

When Ω is finite, then, S_b is thus a finitely-generated cone, and is therefore closed.

□

We can now prove Theorem 3.14.

Proof of Theorem 3.14. The proof follows that of Theorem 3.10. Clearly $\mathcal{D} \subseteq \mathcal{D}_{\mathcal{F}_\mathcal{D}}$.

If $b \in \mathcal{D}_{\mathcal{F}_\mathcal{D}}$ then there are $a_1, \ldots, a_n \in \mathcal{D}$ with $[\![\Pr(b) > 0]\!] \supseteq [\![\Pr(a_1) > 0]\!] \cap \ldots \cap [\![\Pr(a_n) > 0]\!]$. By Sublemma 3.10.3, this entails that $\mathsf{conv}(\{a_1, \ldots, a_n\})$ and $\mathsf{closure}(S_b)$ are non-disjoint. By Sublemma A.1.2, S_b is closed, ensuring that in fact $\mathsf{conv}(\{a_1, \ldots, a_n\})$ and S_b are non-disjoint. Then since $a_1, \ldots, a_n \in \mathcal{D}$, by axioms ($\mathrm{D}_\lambda$) and ($\mathrm{D}_+$), $\mathcal{D} \cap S_b \neq \varnothing$, so by Sublemma A.1.1, $b \in \mathcal{D}$, as required.

To show that $\mathcal{F}_\mathcal{D}$ is coherent: It is closed under finite intersection and supersets by definition. By axiom ($\mathrm{D}_{\inf > 0}$), $\mathcal{D} \neq \varnothing$, so $\mathcal{F}_\mathcal{D} \neq \varnothing$. To check that it is proper, observe that $\varnothing = [\![\Pr(0) > 0]\!] \notin \mathcal{F}_\mathcal{D}$, making use of axiom ($\mathrm{D}_{0\notin}$). □

A.3 Proof of Theorem 3.17

In this case we cannot prove that S_b is closed. Instead we make use of axiom ($\mathrm{D}_{\mathrm{ArchCl}}$) to move directly from non-disjointness with $\mathsf{closure}(S_b)$ to that $b \in \mathcal{D}$. We first need a few subsidiary lemmas.

Sublemma A.1.3. *Suppose $\mathcal{D}$ satisfies axioms* ($\mathrm{D}_{0\notin}$)*,* (D_λ) *and* ($\mathrm{D}_{\mathrm{ArchCl}}$)*. If $\mathcal{D} \cap \mathsf{closure}(S_b) \neq \varnothing$, then $b \in \mathcal{D}$.*

Proof. As in Sublemma 3.10.2, $S_b = (\mathsf{posi}(\{b\}) - \mathcal{G}_{\geq 0}) \cup \mathcal{G}_{\leq 0}$. So $\mathsf{closure}(S_b) = \mathsf{closure}(\mathsf{posi}(\{b\}) - \mathcal{G}_{\geq 0}) \cup \mathcal{G}_{\leq 0}$.

Our axioms entail that $\mathcal{D} \cap \mathcal{G}_{\leq 0} = \varnothing$. Otherwise, suppose $f \in \mathcal{D} \cap \mathcal{G}_{\leq 0}$. So $f \leq 0$. By axiom ($\mathrm{D}_\geq$), then $0 \in \mathcal{D}$ contradicting axiom ($\mathrm{D}_{0\notin}$).

Thus in fact $\mathcal{D} \cap \mathsf{closure}(\mathsf{posi}(\{b\}) - \mathcal{G}_{\geq 0}) \neq \varnothing$. Call a member of it f^*. Thus, for each $\epsilon > 0$ there is $s_\epsilon \in \mathsf{posi}(\{b\}) - \mathcal{G}_{\geq 0}$ with $\|s_\epsilon - f^*\|_\infty < \epsilon$.

Observe then that $s_\epsilon(\omega) - f^*(\omega) \geq -\epsilon$ for all $\omega \in \Omega$, so $s_\epsilon - f^* + \epsilon \in \mathcal{G}_{\geq 0}$.

We have that $s_\epsilon \in \mathsf{posi}(\{b\}) - \mathcal{G}_{\geq 0}$ so put $s_\epsilon = \lambda b + h$ with $\lambda > 0$ and $h \in \mathcal{G}_{\geq 0}$. Then

$$\begin{aligned} b &= {}^1\!/\!_\lambda s_\epsilon + h \\ &= {}^1\!/\!_\lambda (f^* - \epsilon) + {}^1\!/\!_\lambda (s_\epsilon - f^* + \epsilon) + h \end{aligned}$$

with ${}^1\!/\!_\lambda (s_\epsilon - f^* + \epsilon) + h \in \mathcal{G}_{\geq 0}$, so $b \in \mathsf{posi}(\{f^* - \epsilon\}) + \mathcal{G}_{\geq 0}$ for all $\epsilon > 0$. Since $f^* \in \mathcal{D}$, by axiom ($\mathrm{D}_{\mathrm{ArchCl}}$), we can conclude that $b \in \mathcal{D}$. □

We can now prove the main result.

Proof of Theorem 3.17. The proof follows that of Theorem 3.10. Assume $\mathcal{D}$ satisfies axioms ($\mathrm{D}_{0\notin}$), (D_λ), (D_+) ($\mathrm{D}_{\inf>0}$), ($\mathrm{D}_\geq$) and ($\mathrm{D}_{\mathrm{ArchCl}}$), Clearly $\mathcal{D} \subseteq \mathcal{D}_{\mathcal{F}_\mathcal{D}}$.

If $b \in \mathcal{D}_{\mathcal{F}_\mathcal{D}}$ then there are $a_1, \ldots, a_n \in \mathcal{D}$ with $[\![\Pr(b) > 0]\!] \supseteq [\![\Pr(a_1) > 0]\!] \cap \ldots \cap [\![\Pr(a_n) > 0]\!]$. By Sublemma 3.10.3, this entails that $\mathsf{conv}(\{a_1, \ldots, a_n\})$ and $\mathsf{closure}(S_b)$ are non-disjoint. Let $f \in \mathsf{conv}(\{a_1, \ldots, a_n\}) \cap \mathsf{closure}(S_b)$. By axioms ($\mathrm{D}_\lambda$) and ($\mathrm{D}_+$) then $f \in \mathcal{D}$, so $\mathcal{D} \cap \mathsf{closure}(S_b) \neq \varnothing$, entailing that $b \in \mathcal{D}$ by Sublemma A.1.3.

The proof that $\mathcal{F}_\mathcal{D}$ is coherent is as in the case when Ω is finite. □

A.4 Proof of claims in Example 3.16

Proof. For any $\lambda > 0$, when $n > {}^1\!/\!_\lambda$ then ${}^1\!/\!_{n^2} < \lambda {}^1\!/\!_n$, i.e., $b(n) < \lambda a(n)$. Thus for no $\lambda > 0$ do we have $b \geq \lambda a$, as this would require that $b(n) \geq \lambda a(n)$ for all n. So $b \notin \mathsf{posi}(\{a\}) + \mathcal{G}_{\geq 0}$.

An analogous argument shows that $a \notin S_b = \mathsf{cone}(\{b\}) - \mathcal{G}_{\geq 0}$. For any $\mu \geq 0$ and $n > \mu$, then $a(n) > \mu b(n)$, so for no $\mu \geq 0$ do we have $a \leq \mu b$, ensuring that $a \notin S_b$.

We will show that $b \in \mathsf{posi}(\{a - \epsilon\}) + \mathcal{G}_{\geq 0}$ for all $\epsilon > 0$. Consider some $\epsilon > 0$. We will find some $\lambda_\epsilon > 0$ such that $b(n) \geq \lambda_\epsilon (a(n) - \epsilon)$ for all n.

For $n \geq {}^1\!/\!_\epsilon$, $a(n) - \epsilon = {}^1\!/\!_n - \epsilon \leq 0$, so $b(n) \geq 0 \geq \lambda(a(n) - \epsilon)$ for any $\lambda > 0$.

Also for $n = 0$, $a(0) - \epsilon < 0$, so $b(0) = 0 \geq a(0) - \epsilon$.

For $0 < n < {}^1\!/\!_\epsilon$, ${}^1\!/\!_n - \epsilon > 0$. So consider $\lambda_\epsilon := \min\{\frac{1/n^2}{1/n - \epsilon} \mid n < {}^1\!/\!_\epsilon\} > 0$, since this is a finite set of positive reals. For any $n < {}^1\!/\!_\epsilon$, $\lambda_\epsilon \leq \frac{1/n^2}{1/n - \epsilon}$ so ${}^1\!/\!_{n^2} \geq \lambda_\epsilon({}^1\!/\!_n - \epsilon)$, i.e., $b(n) \geq \lambda_\epsilon(a(n) - \epsilon)$ for $n < {}^1\!/\!_\epsilon$.

We thus have $b(n) \geq \lambda_\epsilon(a(n) - \epsilon)$ for all n, i.e., $b \geq \lambda_\epsilon(a - \epsilon)$; giving us $b \in \mathsf{posi}(\{a - \epsilon\}) + \mathcal{G}_{\geq 0}$. This also shows us that $a - \epsilon \in S_b = \mathsf{cone}(\{b\}) - \mathcal{G}_{\geq 0}$. □